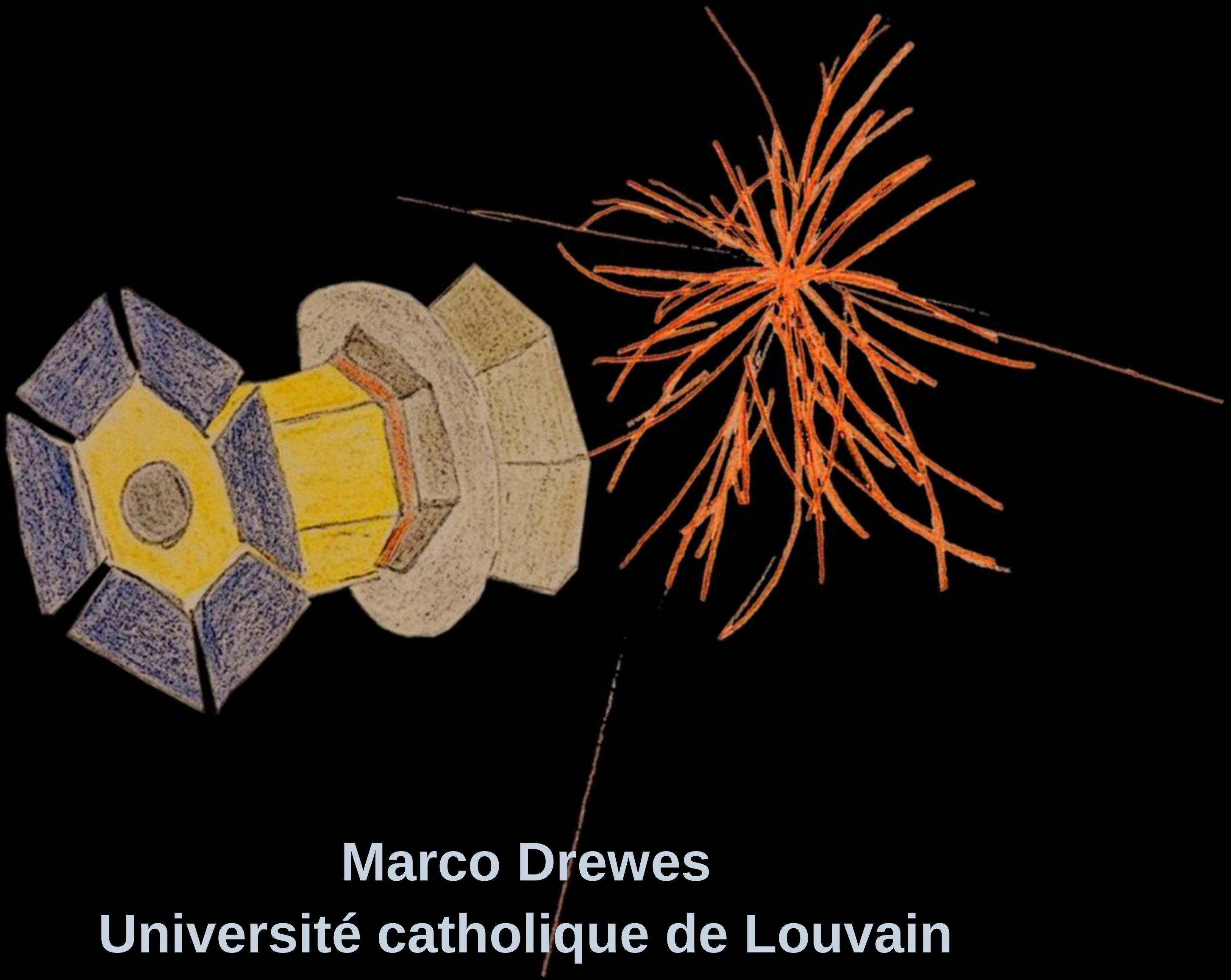
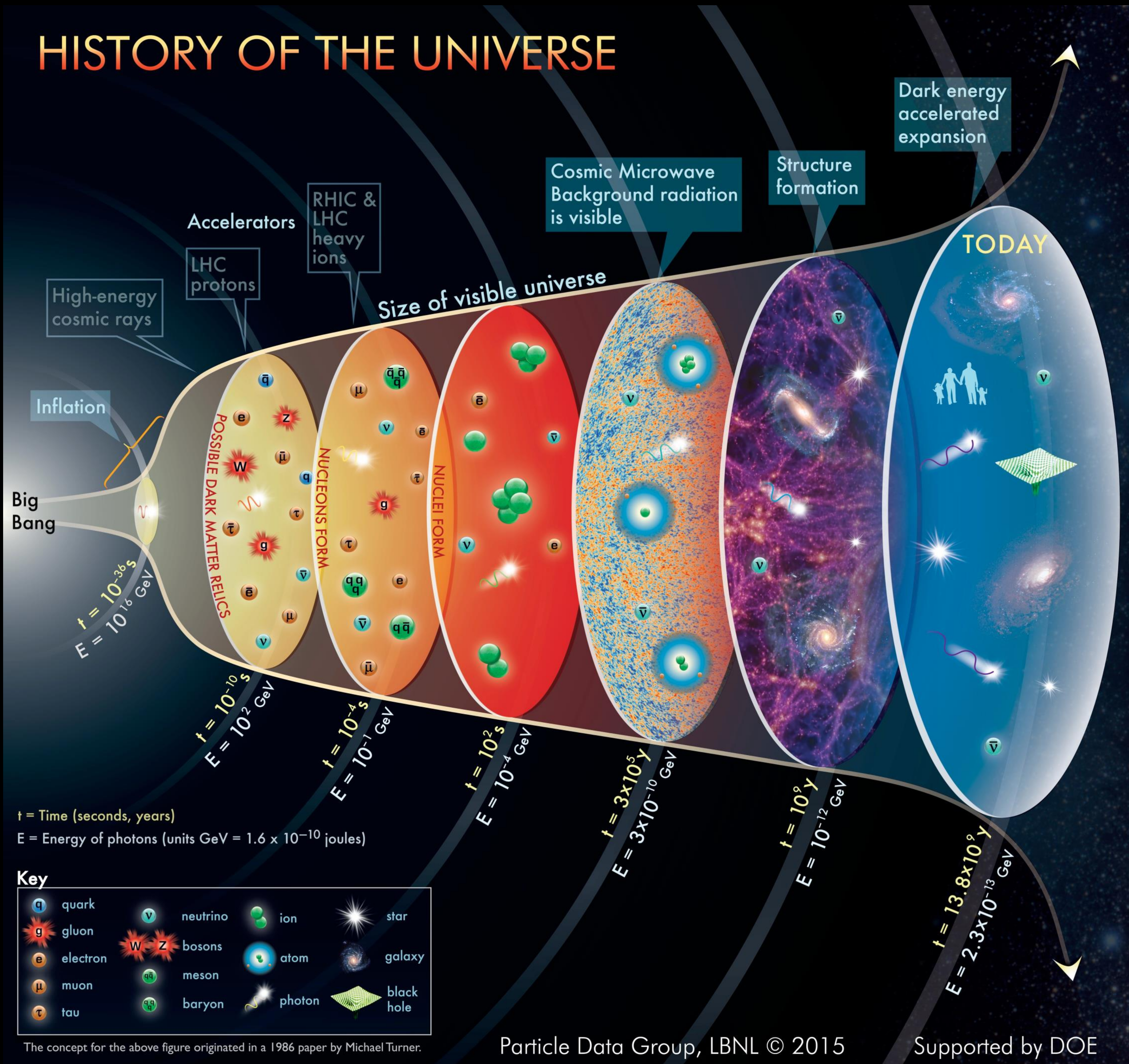
Neutrino Decoupling in Nonequilibrium QFT



Marco Drewes
Université catholique de Louvain

EuCAPT Astroparticle Neutrino Workshop 2024, Prague, Czech Republic


HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Structure formation
Cosmic Microwave Background radiation is visible
RHIC & LHC heavy ions
Accelerators
LHC protons
High-energy cosmic rays
Size of visible universe
Inflation
TODAY
Big Bang
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
$t = 10^{-36}$ s
$E = 10^{16}$ GeV
$t = 10^{-10}$ s
$E = 10^{2}$ GeV
$t = 10^{-4}$ s
$E = 10^{-1}$ GeV
$t = 10^{2}$ s
$E = 10^{-4}$ GeV
$t = 3\text{x}10^{5}$ y
$E = 3\text{x}10^{-10}$ GeV
$t = 10^{9}$ y
$E = 10^{-12}$ GeV
$t = 13.8\text{x}10^{9}$ y
$E = 2.3\text{x}10^{-13}$ GeV
t = Time (seconds, years)
E = Energy of photons (units GeV = 1.6×10^{-10} joules)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
The concept for the above figure originated in a 1986 paper by Michael Turner.
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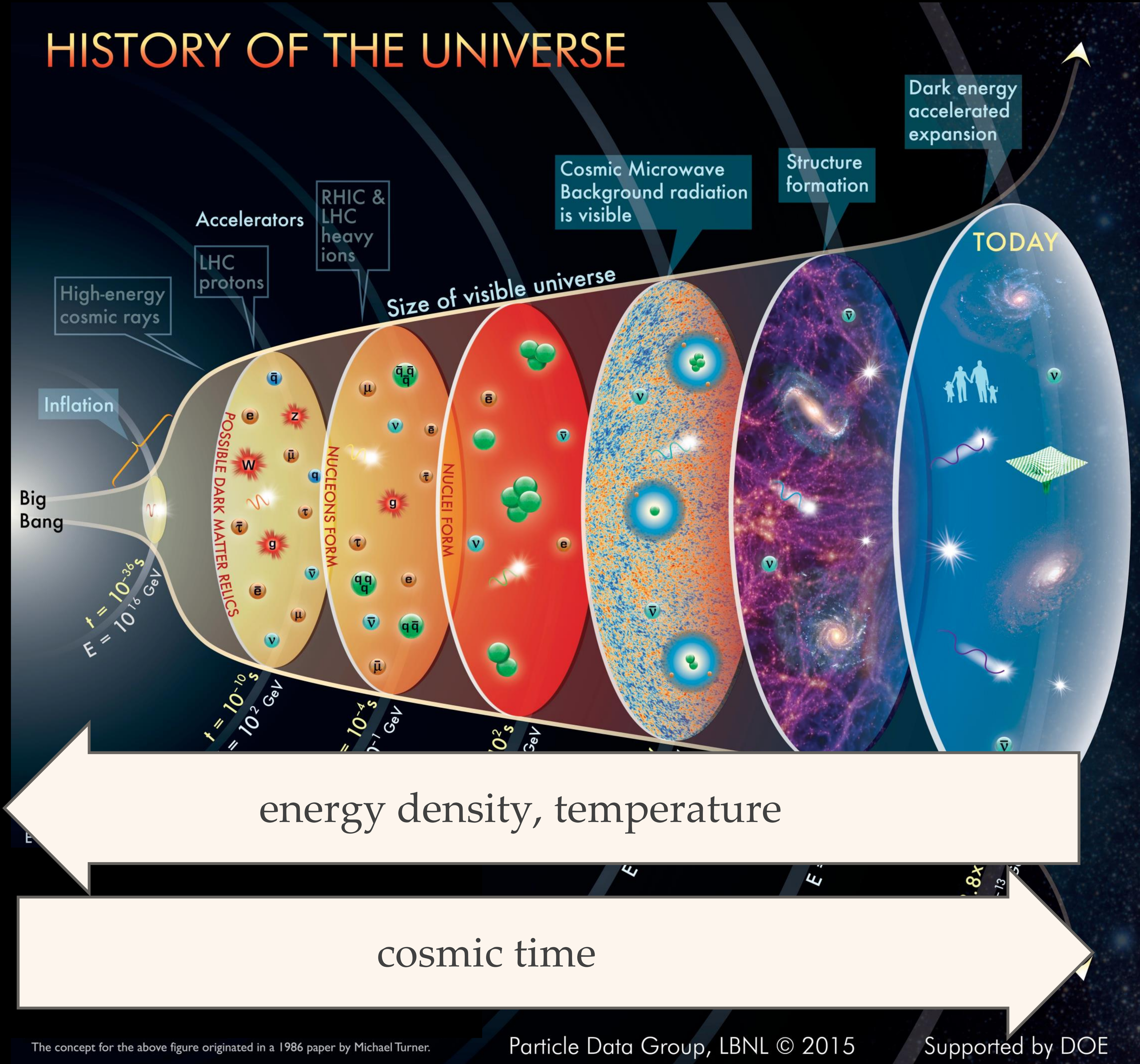
HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
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Inflation
Size of visible universe
TODAY
Big Bang
t = 10^-36 s
E = 10^16 GeV
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
t = 10^-10 s
energy density, temperature
cosmic time
The concept for the above figure originated in a 1986 paper by Michael Turner.
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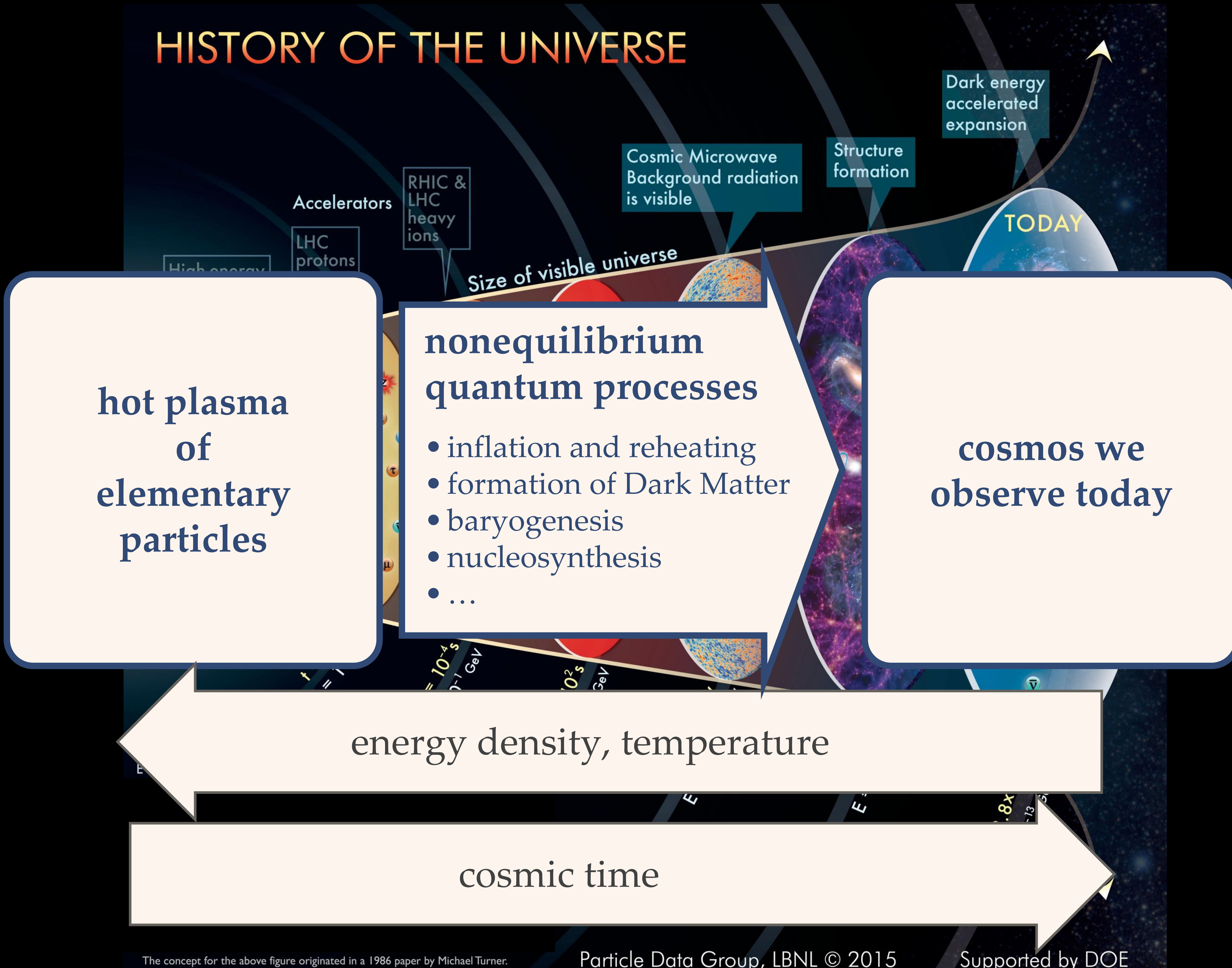
HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Cosmic Microwave Background radiation is visible
Structure formation
Accelerators
RHIC & LHC heavy ions
LHC protons
Size of visible universe
TODAY
hot plasma of elementary particles
nonequilibrium quantum processes
• inflation and reheating
• formation of Dark Matter
• baryogenesis
• nucleosynthesis
• …
cosmos we observe today
energy density, temperature
cosmic time
The concept for the above figure originated in a 1986 paper by Michael Turner.
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Boltzmann Equations

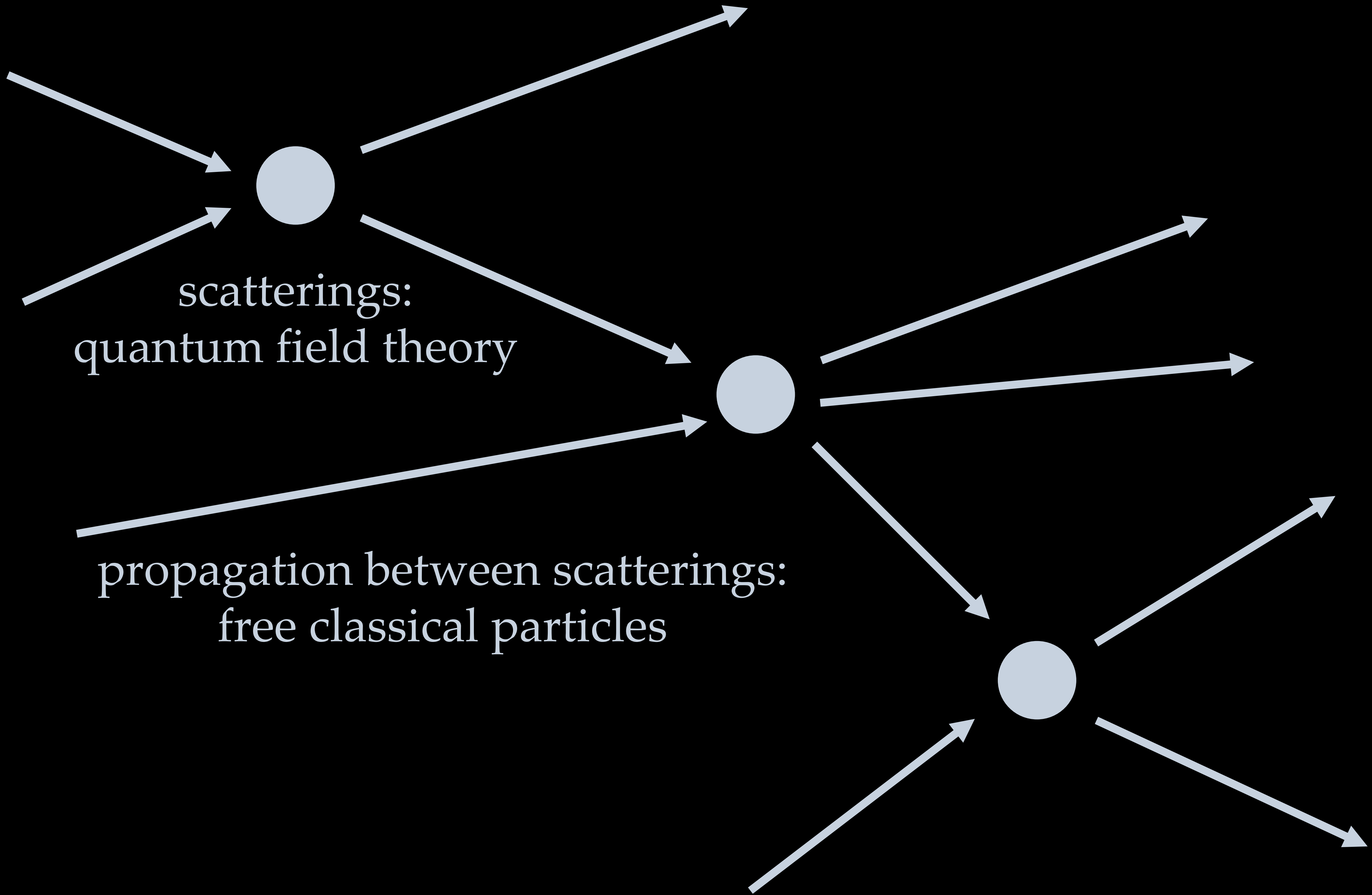
- Equation of motion for classical phase space density

$$\left[\omega_i \frac{\partial}{\partial t} - H \mathbf{p}_i^2 \frac{\partial}{\partial \omega_i}\right] f_i = \mathcal{I}[\{f_j\}]$$

- Collision term imported from QFT in vacuum

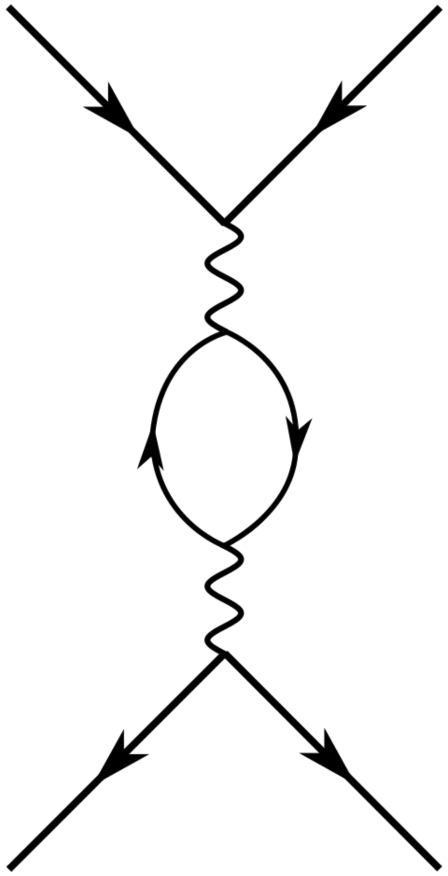
$$\begin{aligned}
\mathcal{I}[\{f_j\}] \; = \; & -\frac{1}{2}\int \left(\frac{g_a\, d^3\mathbf{p}_a}{(2\pi)^3\; 2\omega_a}\right)\left(\frac{g_b\, d^3\mathbf{p}_b}{(2\pi)^3\; 2\omega_b}\right)\cdots\left(\frac{g_u\, d^3\mathbf{p}_u}{(2\pi)^3\; 2\omega_u}\right)\left(\frac{g_v\, d^3\mathbf{p}_v}{(2\pi)^3\; 2\omega_v}\right)\cdots \\
& \times\; (2\pi)^2 \delta(p_i + p_a + p_b \ldots - p_u - p_v \ldots) \\
& \times\; \Big[|\mathcal{M}|^2_{i+a+b\ldots\to u+v+\ldots}\; f_i f_a f_b \ldots\; (1 \pm f_u)(1 \pm f_v)\ldots \\
& \quad |\mathcal{M}|^2_{u+v\ldots\to i+a+b+\ldots}\; f_u f_v \ldots\; (1 \pm f_i)(1 \pm f_a)(1 \pm f_b)\ldots \Big]
\end{aligned}$$

Boltzmann Equations



Shortcomings of Boltzmann Equations

- picture the early universe as a series of collider events
- Often works well…. but misses a few crucial aspects:



$$\left[\omega_i \frac{\partial}{\partial t} - H \mathbf{p}_i^2 \frac{\partial}{\partial \omega_i}\right] f_i = \mathcal{I}[\{f_j\}]$$

- coherence and decoherence
- quantum statistical effects on internal propagators
- screening of particles into quasiparticles
- collective excitations of the plasma
- multiple coherent scatterings
- non-perturbative effects

- "first principles" description makes sure we do not miss these
- At the same time: Need equations that are simple enough for parameter space scans

⇒ Quantum Kinetic Theory to make predictions for accelerator experiments

Shortcomings of Boltzmann Equations

- Some of the shortcomings can be overcome with density matrix equations

$$\partial_t \rho - p H \partial_p \rho = -i[\mathbb{H}, \rho] + \mathcal{I}[\rho].$$

Raffelt/Sigl 1992

$$\mathcal{I}[\rho] = \frac{1}{2}\Big((1-\rho)\Gamma^{<} - \rho\Gamma^{>}\Big) + \text{h.c.}$$

- In the early universe we in addition need the the continuity equation

$$\frac{d\rho_{\text{tot}}}{dt} + 3H(\rho_{\text{tot}} + P_{\text{tot}}) = 0,$$

- And of course the Friedmann equation

$$H^2 = \rho_{\text{tot}}/(3M_{pl}^2)$$

Quantitative Description

- Full information about quantum statistical system contained in von Neumann density operator, with equation of motion

$$\dot{\varrho} = -\mathrm{i}[H, \varrho]$$

- Equivalently: consider infinite tower of n-point functions with expectation values

$$\langle \ldots \rangle = \mathrm{Tr}(\varrho \ldots)$$

- in practice usually one- and two-point functions are sufficient
- Expressing all observables in terms of correlation functions avoids semi-classical assumptions or reference to asymptotic states
- Equations of motion obtained from 2PI effective action in the Schwinger-Keldysh formalism (e.g. Kadanoff-Baym equations); usually non-Markovian and not suitable for parameter scans
- Obtain effective quantum kinetic equations suitable for numerics in a series of controlled approximations adapted to the problem under consideration (gradient expansion in Wigner space, loop truncation, quasiparticle approximation…)

Literature

Equilibrium and Nonequilibrium Formalisms Made Unified #2

Kuang-chao Chou (Beijing, Inst. Theor. Phys.), Zhao-bin Su (Beijing, Inst. Theor. Phys.), Bai-lin Hao (Beijing, Inst. Theor. Phys.), Lu Yu (Beijing, Inst. Theor. Phys.) (Jun, 1984)

Published in: *Phys.Rept.* 118 (1985) 1-131

Introduction to nonequilibrium quantum field theory #2

Juergen Berges (Heidelberg U.) (Sep, 2004)

Published in: *AIP Conf.Proc.* 739 (2004) 1, 3-62 • Contribution to: 9th Hadron Physics and 7th Relativistic Aspects of Nuclear Physics (HADRON-RANP 2004): A Joint Meeting on QCD and QGP, 3-62 • e-Print: hep-ph/0409233 [hep-ph]

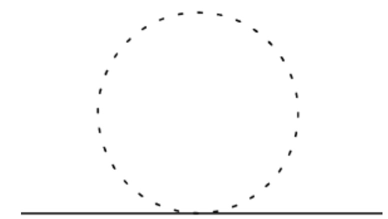
Why is there more matter than antimatter? Calculational methods for leptogenesis and electroweak baryogenesis #1

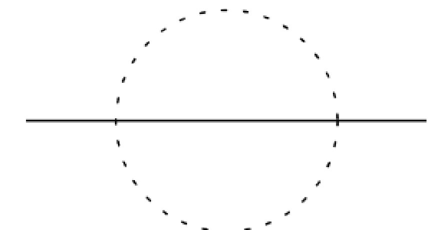
Björn Garbrecht (Munich, Tech. U.) (Dec 6, 2018)

Published in: *Prog.Part.Nucl.Phys.* 110 (2020) 103727 • e-Print: 1812.02651 [hep-ph]

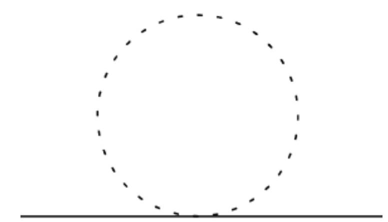
Models

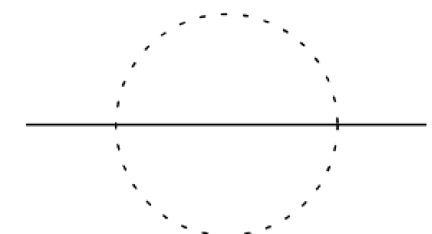
$$\mathcal{L}_{\rm NC}^{(e\nu)} = -\frac{G_F}{\sqrt{2}}[\bar{\nu}_\alpha\,\gamma_\mu(1-\gamma_5)\,\nu_\alpha][\bar{e}(p')\,\gamma^\mu(g_V - g_A\gamma_5)\,e],$$



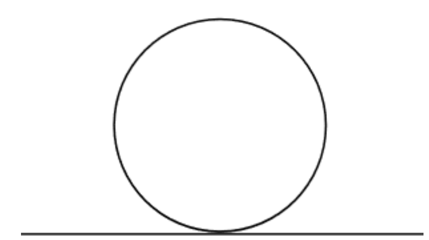


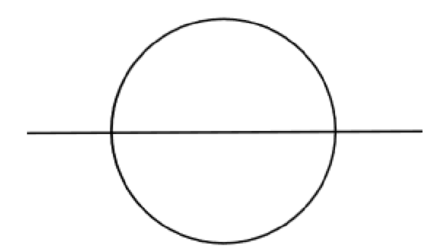
$$\mathcal{L}_{\rm CC}^{(e\nu)} = -\frac{G_F}{\sqrt{2}}[\bar{\nu}_e\,\gamma_\mu(1-\gamma_5)\,e][\bar{e}\,\gamma^\mu(1-\gamma_5)\,\nu_e]$$





$$\mathcal{L}_{NC}^{(\nu\nu)} = -\frac{G_F}{\sqrt{2}}[\bar{\nu}_\alpha\,\gamma_\mu(1-\gamma_5)\,\nu_\alpha][\bar{\nu}_\beta\,\gamma^\mu(1-\gamma_5)\,\nu_\beta]$$





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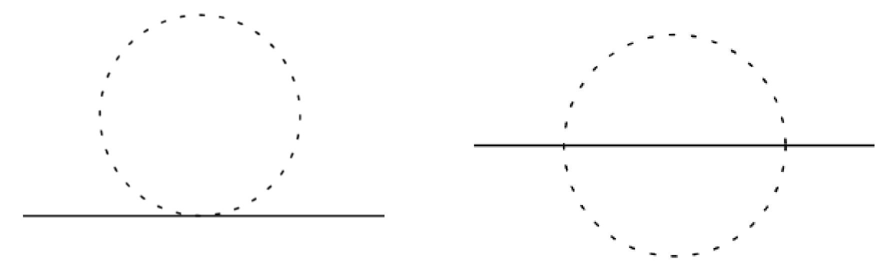

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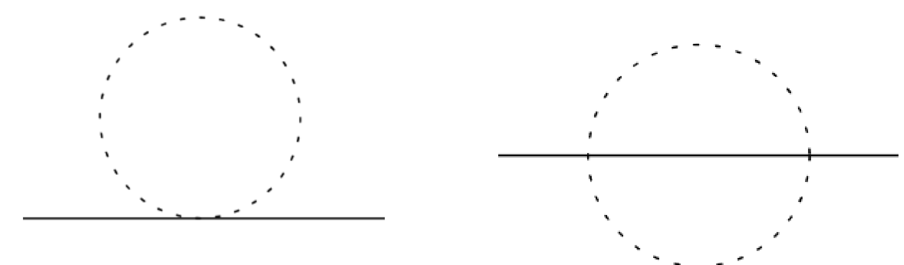

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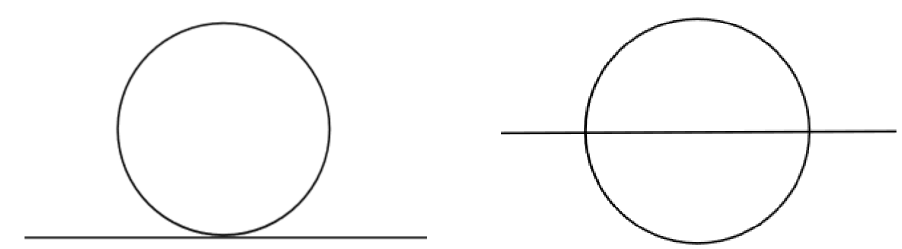
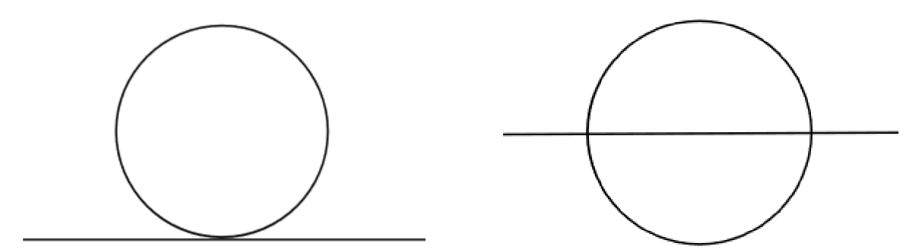


$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4$$



Correlation Functions

- One point function ("condensate") ~ classical field $\varphi(x) \equiv \langle \Phi(x) \rangle$
- Two independent two-point functions, e.g. the Wightman functions

$$\Delta^{>}(x_1, x_2) \;\; = \;\; \langle \Phi(x_1)\Phi(x_2) \rangle - \varphi(x_1)\varphi(x_2)$$

$$\Delta^{<}(x_1, x_2) = \langle \phi(x_2)\phi(x_1) \rangle - \varphi(x_1)\varphi(x_2)$$

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- Time-ordered (Feynman) propagator

$$\Delta^{F}(x_1, x_2) = \theta(t_1 - t_2)\Delta^{>}(x_1, x_2) + \theta(x_2^0, x_1^0)\Delta^{<}(x_1, x_2)$$

- Anti-time-ordered propagator

$$\Delta^{\bar{F}}(x_1, x_2) = \theta(t_1 - t_2)\Delta^{<}(x_1, x_2) + \theta(x_2^0, x_1^0)\Delta^{>}(x_1, x_2)$$

- Advanced propagator $\mathrm{i}\Delta^{A}(x_1, x_2) = -\theta(t_2 - t_1)\Delta^{-}(x_1, x_2)$
- Retarded propagator $\mathrm{i}\Delta^{R}(x_1, x_2) = \theta(t_1 - t_2)\Delta^{-}(x_1, x_2)$
- Spectral function $\Delta^{-}(x_1, x_2) = \mathrm{i}(\Delta^{>}(x_1, x_2) - \Delta^{<}(x_1, x_2))$
- Statistical propagator $\Delta^{+}(x_1, x_2) = \frac{1}{2}(\Delta^{>}(x_1, x_2) + \Delta^{<}(x_1, x_2))$

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- Spectral function (defines spectrum of quasiparticles)

$$\Delta^{-}(x_1, x_2) = \mathrm{i}(\Delta^{>}(x_1, x_2) - \Delta^{<}(x_1, x_2))$$

- Statistical propagator (generalised occupation numbers)

$$\Delta^{+}(x_1, x_2) = \frac{1}{2}(\Delta^{>}(x_1, x_2) + \Delta^{<}(x_1, x_2))$$

Correlation Functions

- One point function ("condensate") ~ classical field $\varphi(x) \equiv \langle \Phi(x) \rangle$
- Two independent two-point functions, e.g. the Wightman functions
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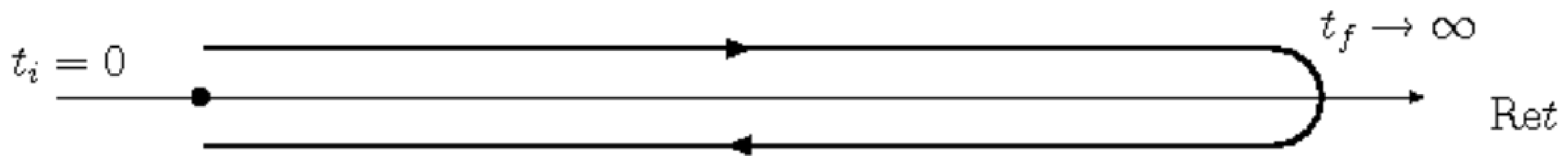
Closed Time Path Formalism

- Want to solve initial value problems, i.e., impose boundary conditions at given time
- S-matrix with projection on asymptotic states in infinite past/future nut ideal tool
- Define correlation functions on “closed time path” (CTP)



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- Define correlation functions on “closed time path” (CTP)

$$\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$$

The following are all basically the same thing:

- Closed time-path formalism
- Schwinger-Keldysh formalism
- In-in formalism
- Real-time formalism (in equilibrium, as opposed to imaginary-time formalism)

Closed Time Path Formalism

- Want to solve initial value problems, i.e., impose boundary conditions at given time
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$\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$

- Define propagator on the contour

$$\Delta_{\mathcal{C}}(x_1, x_2) = \langle \mathrm{T}_{\mathcal{C}} \phi(x_1) \phi(x_2) \rangle = \theta_{\mathcal{C}}(x_1^0, x_2^0) \Delta^{>}(x_1, x_2) + \theta_{\mathcal{C}}(x_2^0, x_1^0) \Delta^{<}(x_1, x_2)$$

Closed Time Path Formalism

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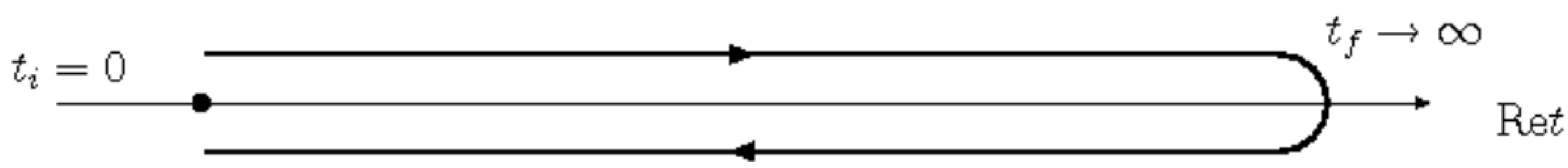

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- Formally consider field with time argument on the "forward" and "backward" parts of the contour like different fields Φ_+ Time argument on forward branch

  Φ_- Time argument on backwards branch

- Promotes propagator to a matrix

$$\begin{pmatrix} \Delta_{++}(x_1, x_2) & \Delta_{+-}(x_1, x_2) \\ \Delta_{-+}(x_1, x_2) & \Delta_{--}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \Delta^F(x_1, x_2) & \Delta^<(x_1, x_2) \\ \Delta^>(x_1, x_2) & \Delta^{\bar{F}}(x_1, x_2) \end{pmatrix}$$

Perturbation Theory

$$\begin{pmatrix} \Delta_{++}(x_1,x_2) & \Delta_{+-}(x_1,x_2) \\ \Delta_{-+}(x_1,x_2) & \Delta_{--}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} \Delta^{F}(x_1,x_2) & \Delta^{<}(x_1,x_2) \\ \Delta^{>}(x_1,x_2) & \Delta^{\bar{F}}(x_1,x_2) \end{pmatrix}$$

- Action is local, hence vertices either only connect "+"-fields or "-"-fields
- Therefore vertices are either "+" or "-"
- But fields can propagate into each other via off-diagonal proparators

Feynman rules

i) Draw all diagrams as you would in standard QFT.

ii) Associate all external ends of propagators with the Keldysh index $+$.

iii) Internal vertices can be of either type, $+$ or $-$. Sum over all over all combinatorical possibilities. The vertices represent the same expressions as they would in standard QFT, but include an overall factor -1 for each $-$-type vertex.

iv) Connect the vertices with the appropriate propagators. The Feynman propagator Δ_F always connects to $+$-type vertices, $\Delta_{\bar{F}}$ always connects to $-$-type vertices, but $\Delta^<$ connects a $+$-type vertex to a $-$-type one (and $\Delta^>$ vice versa).

v) Integrate over all internal positions as usual.

Equations of Motion

- Schwinger-Dyson equation on the contour

$$(\Box_1 + M_{\rm tree}(x)^2)\Delta_{\mathcal{C}}(x_1, x_2) + \int_{\mathcal{C}} d^4x' \Pi_{\mathcal{C}}(x_1, x')\Delta_{\mathcal{C}}(x', x_2) = -i\delta_{\mathcal{C}}(x_1 - x_2)$$

- Tree-level mass is defined via inverse classical propagator

$$\mathrm{i}G_\phi^{-1}[\varphi](x_1, x_2) = -(\Box_{x_1} + M^2_{\rm tree})\delta_{\mathcal{C}}(x_1 - x_2),$$

$$\mathrm{i}G^{-1}[\varphi](x_1, x_2) = \frac{\delta^2 S[\Phi]}{\delta\Phi(x_1)\delta\Phi(x_2)}\Big|_{\Phi\to\varphi}$$

- Split self-energy up according to time argument

$$\Pi_{\mathcal{C}}(x_1, x_2) = -\mathrm{i}\Pi^{\rm loc}(x_1)\,\delta(x_1 - x_2)_{\mathcal{C}} + \theta_{\mathcal{C}}(x_1^0, x_2^0)\Pi^>(x_1, x_2) + \theta_{\mathcal{C}}(x_2^0, x_1^0)\Pi^<(x_1, x_2)\ ,$$

- Define the effective mass

$$M^2 = M^2_{\rm tree} + \Pi^{\rm loc}.$$

Equations of Motion

- Schwinger-Dyson equation on the contour

$$(\Box_1 + M_{\text{tree}}(x)^2)\Delta_{\mathcal{C}}(x_1,x_2) + \int_{\mathcal{C}} d^4x' \Pi_{\mathcal{C}}(x_1,x')\Delta_{\mathcal{C}}(x',x_2) = -i\delta_{\mathcal{C}}(x_1-x_2)$$

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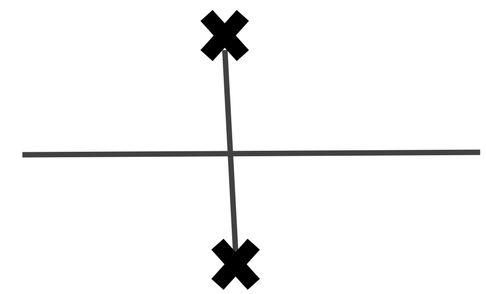

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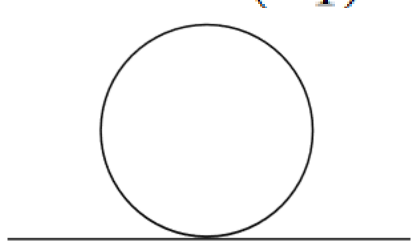

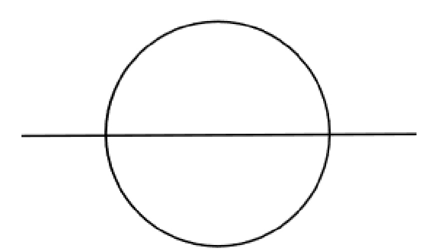

- Define the effective mass

$$M^2 = M_{\text{tree}}^2 + \Pi^{\text{loc}}.$$

Equations of Motion

- Schwinger-Dyson equation on the contour

$$(\Box_1 + M_{\text{tree}}(x)^2)\Delta_{\mathcal{C}}(x_1, x_2) + \int_{\mathcal{C}} d^4x' \Pi_C(x_1, x')\Delta_{\mathcal{C}}(x', x_2) = -i\delta_{\mathcal{C}}(x_1 - x_2)$$

- Consider for instance +- propagator

$$\begin{aligned} & (\Box_1 + M(x_1)^2)\Delta_{+-}(x_1, x_2) + \int_{-\infty}^{\infty} d^4x' \Pi_{++}(x_1, x')\Delta_{+-}(x', x_2) + \int_{\infty}^{-\infty} d^4x' \Pi_{+-}(x_1, x')\Delta_{--}(x', x_2) \\ = \; & (\Box_1 + M(x_1)^2)\Delta_{+-}(x_1, x_2) \int_{-\infty}^{\infty} d^4x' \Pi_{++}(x_1, x')\Delta_{+-}(x', x_2) - \int_{-\infty}^{\infty} d^4x' \Pi_{+-}(x_1, x')\Delta_{--}(x', x_2) = 0 \end{aligned}$$

- Kadanoff-Baym Equations

$$\begin{aligned} (\Box_1 + M^2)\Delta^<(x_1, x_2) &= \int d^4x' (-\Pi_{++}(x_1, x')\Delta^<(x', x_2) + \Pi^<(x_1, x')\Delta_{--}(x', x_2)) \\ (\Box_1 + M^2)\Delta^>(x_1, x_2) &= \int d^4x' (-\Pi^>(x_1, x')\Delta_{++}(x', x_2) + \Pi_{--}(x_1, x')\Delta^>(x', x_2)) \end{aligned}$$

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$$(\Box_1 + M^2)\Delta^>(x_1, x_2) = \int d^4x'(-\Pi^>(x_1, x')\Delta_{++}(x', x_2) + \Pi_{--}(x_1, x')\Delta^>(x', x_2))$$

- or

$$(\Box_1 + M^2)\Delta^{\gtrless}(x_1, x_2) = -\int d^4x'\big(\Pi^{\gtrless}(x_1, x')\Delta^A(x', x_2) + \Pi^R(x_1, x')\Delta^{\gtrless}(x', x_2)\big)$$

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$$\left(\Box_1 + M^2\right)\Delta^{\gtrless}(x_1,x_2) - \int d^4x'\left(\Pi^H(x_1,x')\Delta^{\gtrless}(x',x_2) + \Pi^{\gtrless}(x_1,x')\Delta^H(x',x_2)\right)$$
$$= \tfrac{1}{2}\int d^4x'(\Pi^>(x_1,x')\Delta^<(x',x_2) - \Pi^<(x_1,x')\Delta^>(x',x_2))$$

with $\Delta^H = \frac{1}{2}(\Delta^A + \Delta^R)$.

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- In equilibrium correlation functions only depend on the relative coordinates (it's static, homogeneous, isotropic)
- The von Neumann density matrix is $\rho = e^{-\mathbb{H}/T}$

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$$\Delta^<(t + \mathrm{i}/T, \mathbf{x}) = \Delta^>(t, \mathbf{x}),$$

- And in momentum space

$$\Delta^<(p_0, \mathbf{p}) = \Delta^>(-p_0, \mathbf{p}) = e^{-p_0/T}\Delta^>(p_0, \mathbf{p}),$$

- This implies the Kubo-Martin-Schwinger (KMS) relations

$$\Delta^<(p) = f_B(p_0)\rho(p) \ , \quad \Delta^>(p) = (1 + f_B(p_0))\rho(p)$$

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- Explains the interpretation of the statistical propagator in terms of occupation numbers!
- Relations also allow to find all free propagators (since we already know the free spectral function) $\rho(p) = 2\pi \mathrm{sign}(p_0)\delta(p^2 - m^2)$

$$\Delta_{++}(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + f_B(|p_0|)2\pi\delta(p^2 - m^2) \quad \Delta_{+-} = (f_B(|p_0|) + \theta(-p_0))2\pi\delta(p^2 - m^2)$$

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Quasiparticles

- In thermal equilibrium we can also find the full spectral function

$$\rho(p) = \frac{-2\mathrm{Im}\mathit{\Pi}^R(p) + 2p_0\epsilon}{[p_0^2 - \Omega_{\mathbf{p}}^2 - \mathrm{Re}\mathit{\Pi}^R(p)]^2 + [\mathrm{Im}\mathit{\Pi}^R(p) - p_0\epsilon]^2}.$$

- With the retarded self-energy

$$\mathit{\Pi}^R(x_1, x_2) = \theta(t_1 - t_2)(\mathit{\Pi}^>(x_1, x_2) - \mathit{\Pi}^<(x_1, x_2)).$$

- Its poles determine the dispersion relations for quasiparticles in the plasma. Consider the pole $\hat{\mathit{\Omega}}_{\mathbf{p}}$

$$\mathit{\Omega}_{\mathbf{p}} = \mathrm{Re}\,\hat{\mathit{\Omega}}_{\mathbf{p}} \qquad \mathit{\Gamma}_{\mathbf{p}} = 2\,\mathrm{Im}\,\hat{\mathit{\Omega}}_{\mathbf{p}}$$

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$$\Omega_{\mathbf{p}} = \mathrm{Re}\,\hat{\Omega}_{\mathbf{p}} \qquad \Gamma_{\mathbf{p}} = 2\,\mathrm{Im}\,\hat{\Omega}_{\mathbf{p}}$$

- We obtain the dispersion relation (real part of the refractive index) by solving

$$p_0^2 - \mathbf{p}^2 - M^2 - \mathrm{Re}\Pi^R(p) = 0,$$

- Then obtain width (imaginary part of refractive index) from

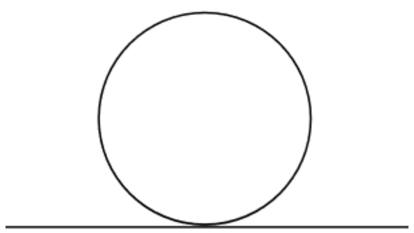
$$\Gamma_{\mathbf{p}} = -\frac{\mathcal{Z}}{\Omega_{\mathbf{p}}}\,\mathrm{Im}\,\Pi^R(p)\Big|_{p_0=\Omega_{\mathbf{p}}}$$

- Near the pole this gives Breit-Wigner approximation

$$\rho(p) \simeq \mathcal{Z}\frac{2\Gamma p_0}{(p_0^2 - \Omega_{\mathbf{p}})^2 + (\Gamma p_0)^2}$$

Matter Potential

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4$$



$$\Pi^{\rm loc} = \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\Delta_{++}(p)$$

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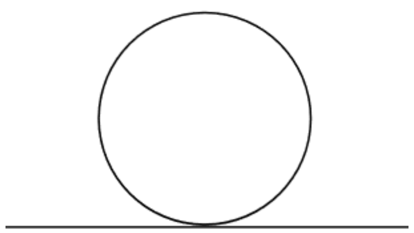

$$\Pi^{\mathrm{loc}} = \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\Delta_{++}(p) \Longrightarrow \frac{\lambda}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{f_B(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \simeq \frac{\lambda}{24}T,$$

$$\Delta_{++}(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + f_B(|p_0|)2\pi\delta(p^2 - m^2) \qquad \Delta_{+-} = (f_B(|p_0|) + \theta(-p_0))2\pi\delta(p^2 - m^2)$$

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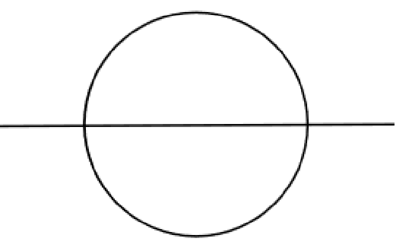
$$M^2 \simeq m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{24}T^2$$

Mean Free Path

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!}\Phi^4$$



$$\Pi_{ab}(p) \;\; = \;\; -ab\frac{\lambda^2}{6}\int \frac{d^4q d^4k d^4l}{(2\pi)^9} \Delta_{ab}(q)\Delta_{ba}(k)\Delta_{ab}(l)\delta(p-q+k-l).$$

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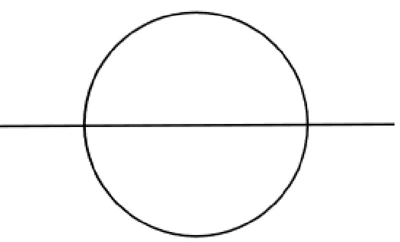

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$$\Gamma_{\mathbf{p}} = -\frac{\mathcal{Z}}{\Omega_{\mathbf{p}}}\,\mathrm{Im}\,\Pi^R(p)\Big|_{p_0=\Omega_{\mathbf{p}}}$$

$$\mathrm{Im}\Pi^R(p) = \frac{1}{2\mathrm{i}}\Pi^-(p) = \frac{1}{2\mathrm{i}} f_B^{-1}(p_0)\Pi^<(p)$$

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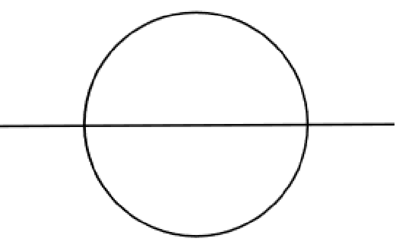

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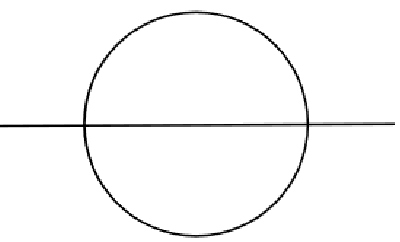

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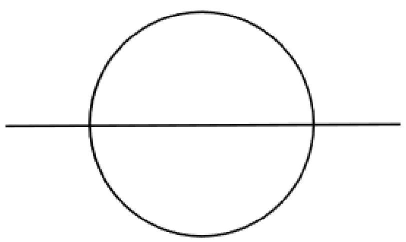
Cutting Rules



$$
\begin{aligned}
\mathrm{Im}\Pi^R(p) = \frac{\lambda^2}{6}\int \frac{d^3\mathbf{q}d^3\mathbf{k}d^3\mathbf{l}}{(2\pi)^9}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{k}+\mathbf{l}-\mathbf{q})\frac{1}{8\omega_{\mathbf{q}}\omega_{\mathbf{k}}\omega_{\mathbf{l}}} \\
\times \Big[((1+f_{\mathbf{q}})(1+f_{\mathbf{k}})(1+f_{\mathbf{l}})-f_{\mathbf{q}}f_{\mathbf{k}}f_{\mathbf{l}}) \\
(\delta(p_0-\omega_{\mathbf{q}}-\omega_{\mathbf{k}}-\omega_{\mathbf{l}})-\delta(p_0+\omega_{\mathbf{q}}+\omega_{\mathbf{k}}+\omega_{\mathbf{l}})) \\
+(f_{\mathbf{q}}(1+f_{\mathbf{k}})(1+f_{\mathbf{l}})-(1+f_{\mathbf{q}})f_{\mathbf{k}}f_{\mathbf{l}})) \\
(\delta(p_0+\omega_{\mathbf{q}}-\omega_{\mathbf{k}}-\omega_{\mathbf{l}})-\delta(p_0-\omega_{\mathbf{q}}+\omega_{\mathbf{k}}+\omega_{\mathbf{l}})) \\
+((1+f_{\mathbf{q}})f_{\mathbf{k}}(1+f_{\mathbf{l}})-f_{\mathbf{q}}(1+f_{\mathbf{k}})f_{\mathbf{l}})) \\
(\delta(p_0-\omega_{\mathbf{q}}+\omega_{\mathbf{k}}-\omega_{\mathbf{l}})-\delta(p_0+\omega_{\mathbf{q}}-\omega_{\mathbf{k}}+\omega_{\mathbf{l}})) \\
+(f_{\mathbf{q}}f_{\mathbf{k}}(1+f_{\mathbf{l}})-(1+f_{\mathbf{q}})(1+f_{\mathbf{k}})f_{\mathbf{l}})) \\
(\delta(p_0+\omega_{\mathbf{q}}+\omega_{\mathbf{k}}-\omega_{\mathbf{l}})-\delta(\omega-\omega_{\mathbf{q}}-\omega_{\mathbf{k}}+\omega_{\mathbf{l}}))\Big]
\end{aligned}
$$

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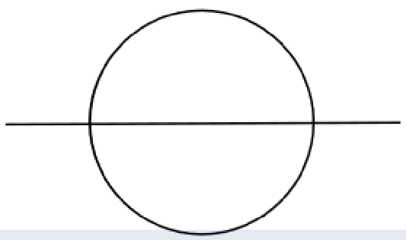
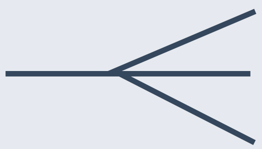
$$\Delta_{-+} = (f_B(|p_0|)+\theta(p_0))2\pi\delta(p^2-m^2) \qquad \Delta_{--}(p) = (\Delta_{++}(p))^*$$

Cutting Rules



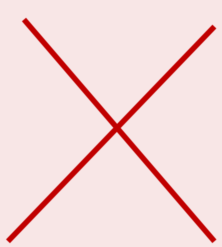
$$\mathrm{Im}\Pi^R(p) = \frac{\lambda^2}{6}\int \frac{d^3\mathbf{q}d^3\mathbf{k}d^3\mathbf{l}}{(2\pi)^9}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{k}+\mathbf{l}-\mathbf{q})\frac{1}{8\omega_\mathbf{q}\omega_\mathbf{k}\omega_\mathbf{l}}$$



Decay an inverse decay

$$\times\Big[((1+f_\mathbf{q})(1+f_\mathbf{k})(1+f_\mathbf{l})-f_\mathbf{q}f_\mathbf{k}f_\mathbf{l})$$
$$(\delta(p_0-\omega_\mathbf{q}-\omega_\mathbf{k}-\omega_\mathbf{l})-\delta(p_0+\omega_\mathbf{q}+\omega_\mathbf{k}+\omega_\mathbf{l}))$$

Various scatterings



$$+(f_\mathbf{q}(1+f_\mathbf{k})(1+f_\mathbf{l})-(1+f_\mathbf{q})f_\mathbf{k}f_\mathbf{l}))$$
$$(\delta(p_0+\omega_\mathbf{q}-\omega_\mathbf{k}-\omega_\mathbf{l})-\delta(p_0-\omega_\mathbf{q}+\omega_\mathbf{k}+\omega_\mathbf{l}))$$
$$+((1+f_\mathbf{q})f_\mathbf{k}(1+f_\mathbf{l})-f_\mathbf{q}(1+f_\mathbf{k})f_\mathbf{l}))$$
$$(\delta(p_0-\omega_\mathbf{q}+\omega_\mathbf{k}-\omega_\mathbf{l})-\delta(p_0+\omega_\mathbf{q}-\omega_\mathbf{k}+\omega_\mathbf{l}))$$
$$+(f_\mathbf{q}f_\mathbf{k}(1+f_\mathbf{l})-(1+f_\mathbf{q})(1+f_\mathbf{k})f_\mathbf{l}))$$
$$(\delta(p_0+\omega_\mathbf{q}+\omega_\mathbf{k}-\omega_\mathbf{l})-\delta(\omega-\omega_\mathbf{q}-\omega_\mathbf{k}+\omega_\mathbf{l}))\Big]$$

$$\Delta_{++}(p)=\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}+f_B(|p_0|)2\pi\delta(p^2-m^2) \qquad \Delta_{+-}=(f_B(|p_0|)+\theta(-p_0))2\pi\delta(p^2-m^2)$$
$$\Delta_{-+}=(f_B(|p_0|)+\theta(p_0))2\pi\delta(p^2-m^2) \qquad \Delta_{--}(p)=(\Delta_{++}(p))^*$$

Fermions

- Define Wightman functions as

$$\mathrm{i}S^{>}_{\alpha\beta}(x_1,x_2) = \langle \Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle \;, \quad \mathrm{i}S^{<}_{\alpha\beta}(x_1,x_2) = -\langle \bar{\Psi}_\beta(x_2)\Psi_\alpha(x_1)\rangle$$

- From those obtain retarded and advanced functions

$$\begin{aligned} \mathrm{i}S^R(x_1,x_2) &= 2\theta(t_1-t_2)S^-(x_1,x_2)\,, \\ \mathrm{i}S^A(x_1,x_2) &= -2\theta(t_2-t_1)S^-(x_1,x_2)\,, \\ S^H(x_1,x_2) &= \tfrac{1}{2}\big(S^R(x_1,x_2)+S^A(x_1,x_2)\big) = -\mathrm{i}\,\mathrm{sign}(t_1-t_2)S^-(x_1,x_2) \end{aligned}$$

- As well as spectral and statistical propagators

$$\begin{aligned} S^-(x_1,x_2) &\equiv \tfrac{\mathrm{i}}{2}\big(S^>(x_1,x_2)-S^<(x_1,x_2)\big) \\ S^+(x_1,x_2) &\equiv \tfrac{1}{2}\big(S^>(x_1,x_2)+S^<(x_1,x_2)\big) \end{aligned}$$

- Which fulfil the Kadanoff-Baym equations

$$\begin{aligned} (\mathrm{i}\partial\!\!\!/_{x_1}-M)S^-(x_1,x_2) &= 2\mathrm{i}\int_{t_1}^{t_2}\mathrm{d}t'\int \mathrm{d}^3\mathbf{x}'\,\Sigma^-(x_1,x')S^-(x',x_2) \\ (\mathrm{i}\partial\!\!\!/_{x_1}-M)S^+(x_1,x_2) &= 2\mathrm{i}\int_{t_i}^{t_2}\mathrm{d}t'\int \mathrm{d}^3\mathbf{x}'\,\Sigma^+(x_1,x')S^-(x',x_2) \\ &\quad -2\mathrm{i}\int_{t_i}^{t_1}\mathrm{d}t'\int \mathrm{d}^3\mathbf{x}'\,\Sigma^-(x_1,x')S^+(x',x_2) \end{aligned}$$

Fermion Propagators

- We can again find the free spectral function

$$\rho(p) = 2\pi(\not{p} + m)\text{sign}(p_0)\delta(p^2 - m^2)$$

- The KMS relations this time read

$$S^<(p) = -e^{-p_0/T} S^>(p_0) \qquad S^+(p) = (1 - 2f_F(p_0))\rho(p)$$

$$S^>(\omega) = (1 - f_F(p_0))\rho(p) \;, \quad S^<(p) = -f_F(p_0)\rho(p)$$

- Yielding propagators

$$\mathrm{i}S^F(p) = \frac{\mathrm{i}(\not{p} + M)}{p^2 - M^2 + \mathrm{i}\epsilon} - 2\pi\delta(p^2 - M^2)(\not{p} + M)f_F(|p_0|) \;=\; \gamma^0(\mathrm{i}S^{\bar{F}}(p))^\dagger\gamma^0$$

$$\mathrm{i}S^<(p) = -2\pi\delta(p^2 - M^2)(\not{p} + M)[f_F(|p_0|) - \theta(-p_0)]\,,$$

$$\mathrm{i}S^>(p) = -2\pi\delta(p^2 - M^2)(\not{p} + M)[f_F(|p_0|) - \theta(p_0)].$$

Fermion Propagators

- We can rewrite this as

$$
\begin{aligned}
\mathrm{i}S^{<}(p) &= -2\pi\delta(p^2 - M^2)(\not{p} + M)\left[\theta(p_0) f_{\mathbf{p}} - \theta(-p_0)(1 - \bar{f}_{\mathbf{p}}))\right] , \\
\mathrm{i}S^{>}(p) &= -2\pi\delta(p^2 - M^2)(\not{p} + M)\left[-\theta(p_0)(1 - f_{\mathbf{p}}) + \theta(-p_0)\bar{f}_{\mathbf{p}})\right] , \\
\mathrm{i}S^{F}(p) &= \frac{\mathrm{i}(\not{p} + M)}{p^2 - M^2 + \mathrm{i}\epsilon} - 2\pi\delta(p^2 - M^2)(\not{p} + M)\left[\theta(p_0) f_{\mathbf{p}} + \theta(-p_0)\bar{f}_{\mathbf{p}})\right] , \\
\mathrm{i}S^{\bar{F}}(p) &= -\frac{\mathrm{i}(\not{p} + M)}{p^2 - M^2 - \mathrm{i}\epsilon} - 2\pi\delta(p^2 - M^2)(\not{p} + M)\left[\theta(p_0) f_{\mathbf{p}} + \theta(-p_0)\bar{f}_{\mathbf{p}})\right]
\end{aligned}
$$

- And define the number of particles and antiparticles with given helicity as

$$
f_{\mathbf{p}h} = \int_0^\infty \frac{dp_0}{2\pi} \mathrm{tr}[\gamma^0 P_h S^+(p)] , \quad \bar{f}_{\mathbf{p}h} = -\int_{-\infty}^0 \frac{dp_0}{2\pi} \mathrm{tr}[\gamma^0 P_h S^+(p)]
$$

- Where the helicity projectors are

$$
P_h \equiv \frac{1}{2}(1 + h\hat{k}\gamma^0 \boldsymbol{\gamma}\gamma^5)
$$

Neutrino Matter Potential

$$\mathcal{L}_{NC}^{(\nu\nu)} = -\frac{G_F}{\sqrt{2}}[\bar{\nu}_\alpha \gamma_\mu (1-\gamma_5)\nu_\alpha][\bar{\nu}_\beta \gamma^\mu (1-\gamma_5)\nu_\beta].$$

- Resummed spectral function in analogy to scalar case

$$\rho(p) = \left(\frac{\mathrm{i}}{\not{p} - M - \Sigma^R(p) + \mathrm{i}\epsilon\gamma_0} - \frac{\mathrm{i}}{\not{p} - M - \Sigma^A(p) - \mathrm{i}\epsilon\gamma_0}\right).$$

Neutrino Matter Potential

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- Poles given by

$$\det\left(\not{p} - m - \Sigma^{\mathrm{loc}} - \mathrm{Re}\, \Sigma^R(p)\right) = 0.$$

- Self-energy structure in general (but at one loop no tensor in homogeneous universe)

$$\Sigma = \left(a_L \not{p} + b_L \not{u} + c_L [\not{p}, \not{u}]\right) P_L.$$

- In the relativistic limit the dispersion relation reads $p_0^2 - \mathbf{p}^2 - m^2 + 2 b_L p_0 = 0,$

Neutrino Matter Potential

$$\mathcal{L}_{NC}^{(\nu\nu)} = -\frac{G_F}{\sqrt{2}}[\bar{\nu}_\alpha \gamma_\mu (1-\gamma_5)\,\nu_\alpha][\bar{\nu}_\beta \gamma^\mu (1-\gamma_5)\,\nu_\beta].$$

- Resumed spectral function in analogy to scalar case

$$\rho(p) = \left(\frac{\mathrm{i}}{\not p - M - \Sigma^R(p) + \mathrm{i}\epsilon\gamma_0} - \frac{\mathrm{i}}{\not p - M - \Sigma^A(p) - \mathrm{i}\epsilon\gamma_0}\right).$$

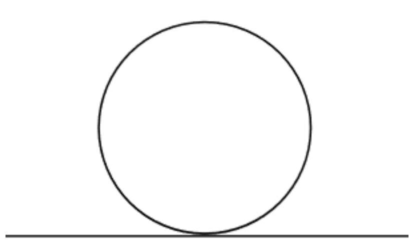
- Poles given by

$$\det\left(\not p - m - \Sigma^{\mathrm{loc}} - \mathrm{Re}\,\Sigma^R(p)\right) = 0.$$

- Self-energy structure in general (but at one loop no tensor in homogeneous universe)

$$\Sigma = \left(a_L \not p + b_L \not u + c_L[\not p, \not u]\right) P_L.$$

- In the relativistic limit the dispersion relation reads $p_0^2 - \mathbf{p}^2 - m^2 + 2b_L p_0 = 0,$



$$\Sigma^{\mathrm{loc}} \supset -\frac{G_F}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma_\mu(1-\gamma_5)\mathrm{i}S^F(k)\right]$$

- Evaluating the integral gives

$$b_L = \pm\sqrt{2}G_F(n_\nu - n_{\bar{\nu}}) \qquad n_\nu = \int \frac{d^3\mathbf{k}}{(2\pi)^3} f(\omega_\mathbf{k})\ , \quad n_{\bar{\nu}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\bar{f}(\omega_\mathbf{k})$$

Density Matrix Equation

- So far we worked in equilibrium. Out of equilibrium we need to solve the KBE.

$$
\begin{aligned}
(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^{-}(x_1,x_2) &= \int \mathrm{d}^4x' \left(\Sigma^{H}(x_1,x')S^{-}(x',x_2) + \Sigma^{-}(x_1,x')S^{H}(x',x_2)\right), \\
(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^{+}(x_1,x_2) &= \int \mathrm{d}^4x' \left(\Sigma^{+}(x_1,x')S^{H}(x',x_2) + \Sigma^{H}(x_1,x')S^{+}(x',x_2)\right) \\
&\quad + \tfrac{1}{2}\int \mathrm{d}^4x' (\Sigma^{>}(x_1,x')S^{<}(x',x_2) - \Sigma^{<}(x_1,x')S^{>}(x',x_2)) \,;
\end{aligned}
$$

- Note that all quantities are matrices in flavour space now

- We use four-spinors throughout, for Majorana neutrinos one simply has an extra condition

$$
S^{\gtrless}(x_1,x_2) = C S^{\gtrless}(x_2,x_1)^{t} C^{\dagger} \,,
$$

Density Matrix Equation

- We slightly rewrite this

$$
\begin{aligned}
(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^-(x_1,x_2) &= \int \mathrm{d}^4x' \left(\Sigma^H(x_1,x')S^-(x',x_2) + \Sigma^-(x_1,x')S^H(x',x_2)\right), \\
(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^+(x_1,x_2) &= \int \mathrm{d}^4x' \left(\Sigma^+(x_1,x')S^H(x',x_2) + \Sigma^H(x_1,x')S^+(x',x_2)\right) \\
&+ \tfrac{1}{2}\int \mathrm{d}^4x' (\Sigma^>(x_1,x')S^<(x',x_2) - \Sigma^<(x_1,x')S^>(x',x_2)).
\end{aligned}
$$

- Analytic solution is impossible… we will perform a gradient expansion
- First we Fourier transform all quantities in the relative coordinate to go to "Wigner space"

$$G(x;k) = \int \mathrm{d}^4 r\, \mathrm{e}^{ikr} G(x + r/2, x - r/2) \qquad x = (x_1 + x_2)/2$$

- The convolutions become very ugly, symbolically

$$\int \mathrm{d}^4(x_1 - x_2) \mathrm{e}^{ik\cdot(x_1-x_2)} \int \mathrm{d}^4 y A(x_1,y)B(y,x_1) = \mathrm{e}^{-\mathrm{i}\diamond}\{A(x;k)\}\{B(x;k)\}$$

$$\diamond\{A\}\{B\} = \frac{1}{2}(\partial_x A \cdot \partial_k B - \partial_k A \cdot \partial_x B)$$

- Luckily the system during neutrino decoupling is very close to equilibrium and adiabatic, so we only need the leading term…

Density Matrix Equation

- We slightly rewrite this

$$(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^-(x_1,x_2) = \int \mathrm{d}^4x' \big(\Sigma^H(x_1,x')S^-(x',x_2) + \Sigma^-(x_1,x')S^H(x',x_2)\big)\,,$$

$$(\mathrm{i}\partial\!\!\!/_{x_1} - M)S^+(x_1,x_2) = \int \mathrm{d}^4x' \big(\Sigma^+(x_1,x')S^H(x',x_2) + \Sigma^H(x_1,x')S^+(x',x_2)\big)$$
$$+\tfrac{1}{2}\int \mathrm{d}^4x' (\Sigma^>(x_1,x')S^<(x',x_2) - \Sigma^<(x_1,x')S^>(x',x_2))\,;$$

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$$G(x;k) = \int \mathrm{d}^4r\, \mathrm{e}^{ikr} G(x + r/2, x - r/2) \qquad x = (x_1 + x_2)/2$$

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$$\diamond\{A\}\{B\} = \frac{1}{2}\left(\partial_x A \cdot \partial_k B - \partial_k A \cdot \partial_x B\right)$$

- Luckily the system during neutrino decoupling is very close to equilibrium and adiabatic…

$$\left(\not{p} + \frac{\mathrm{i}}{2}\gamma_0\partial_t - M\right)S^- - \left(\not{\Sigma}^H S^- + \not{\Sigma}^- S^H\right) = 0\,,$$

$$\left(\not{p} + \frac{\mathrm{i}}{2}\gamma_0\partial_t - M\right)S^+ - \not{\Sigma}^H S^+ - \not{\Sigma}^+ S^H = \tfrac{1}{2}(\not{\Sigma}^> S^< - \not{\Sigma}^< S^>)\,.$$

Density Matrix Equation

$$\left(\not{p} + \frac{\mathrm{i}}{2}\gamma_0\partial_t - M\right)S^- - \left(\not{\Sigma}^H S^- + \not{\Sigma}^- S^H\right) = 0\,,$$

$$\left(\not{p} + \frac{\mathrm{i}}{2}\gamma_0\partial_t - M\right)S^+ - \not{\Sigma}^H S^+ - \not{\Sigma}^+ S^H = \tfrac{1}{2}\left(\not{\Sigma}^> S^< - \not{\Sigma}^< S^>\right).$$

- We define

$$\mathcal{S}^+ \equiv \mathrm{i}\gamma^0 S^+\,,\ \mathcal{S}^H \equiv \mathrm{i}\gamma^0 S^H\,,\ \mathcal{H} \equiv (\not{p} - \not{\Sigma}^H - M)\gamma^0\,,$$

$$\mathcal{G}^> \equiv \not{\Sigma}^>\gamma^0\,,\ \mathcal{G}^< \equiv \not{\Sigma}^<\gamma^0,\ \mathcal{G} \equiv \frac{\mathrm{i}}{2}(\mathcal{G}^> - \mathcal{G}^<),\ \mathcal{N} \equiv \not{\Sigma}^+\gamma^0$$

- Then add and subtract the KBE from their conjugates to obtain “constrained equations” and :kinetic equations”

$$\{\mathcal{H}, \mathcal{S}^-\} - \{\mathcal{G}, \mathcal{S}^H\} = 0$$

$$\mathrm{i}\partial_t\mathcal{S}^- + [\mathcal{H}, \mathcal{S}^-] - [\mathcal{G}, \mathcal{S}^H] = 0$$

$$\{\mathcal{H}, \mathcal{S}^+\} - \{\mathcal{N}, \mathcal{S}^H\} = \tfrac{1}{2}([\mathcal{G}^>, \mathcal{S}^<] - [\mathcal{G}^<, \mathcal{S}^>])\,,$$

$$\mathrm{i}\partial_t\mathcal{S}^+ + [\mathcal{H}, \mathcal{S}^+] - [\mathcal{N}, \mathcal{S}^H] = \tfrac{1}{2}(\{\mathcal{G}^>, \mathcal{S}^<\} - \{\mathcal{G}^<, \mathcal{S}^>\})$$

Density Matrix Equation

- From the kinetic equation we will get the density matrix equation:

$$i\partial_t \mathcal{S}^+ + [\mathcal{H}, \mathcal{S}^+] - [\mathcal{N}, \mathcal{S}^H] = \tfrac{1}{2}(\{\mathcal{G}^>, \mathcal{S}^<\} - \{\mathcal{G}^<, \mathcal{S}^>\})$$

- But we want an equation for on-shell distribution functions, while the above are matrices in spinor space that also exist off-shell. We Lorentz-decompose the propagators

$$\mathcal{S} = \sum_h \frac{1}{2} P_h \left(g_{0h} + \gamma^0 g_{1h} - i\gamma^0\gamma^5 g_{2h} - \gamma^5 g_{3h} \right)$$

- Now we consider the constrained equation

$$\{\mathcal{H}, \mathcal{S}^+\} - \{\mathcal{N}, \mathcal{S}^H\} = \tfrac{1}{2}([\mathcal{G}^>, \mathcal{S}^<] - [\mathcal{G}^<, \mathcal{S}^>])$$

- We multiply it with different combinations of gamma-matrices and take the trace to obtain relations between the Lorentz components

Density Matrix Equation

- From the kinetic equation we will get the density matrix equation:

$$i\partial_t \mathcal{S}^+ + [\mathcal{H}, \mathcal{S}^+] - [\mathcal{N}, \mathcal{S}^H] = \tfrac{1}{2}(\{\mathcal{G}^>, \mathcal{S}^<\} - \{\mathcal{G}^<, \mathcal{S}^>\})$$

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- Now we consider the constrained equation

$$\{\mathcal{H}, \mathcal{S}^+\} - \{\mathcal{N}, \mathcal{S}^H\} = \tfrac{1}{2}([\mathcal{G}^>, \mathcal{S}^<] - [\mathcal{G}^<, \mathcal{S}^>])$$

- We multiply it with different combinations of gamma-matrices and take the trace to obtain relations between the Lorentz components

$$\begin{aligned}
4p_0 g_{0h} &= 4h\mathrm{p} g_{3h} + 2\{g_{1h}, M\} - \{g_{3h} - g_{0h}, b_L + a_L(p_0 + h\mathrm{p})\} \\
4p_0 g_{1h} &= 2\{g_{0h}, M\} + \{g_{1h}, b_L + a_L(p_0 + h\mathrm{p})\} + i[g_{2h}, b_L + a_L(p_0 + h\mathrm{p})] \\
4p_0 g_{3h} &= 4h\mathrm{p} g_{0h} + 2i[g_{0h}, M] + \{g_{3h} - g_{0h}, b_L + a_L(p_0 + h\mathrm{p})\} \\
4p_0 g_{2h} &= 2i[g_{3h}, M] + \{g_{2h}, b_L + a_L(p_0 + h\mathrm{p})\} + i[g_{1h}, b_L + a_L(p_0 + h\mathrm{p})]
\end{aligned}$$

- OK, let's expand in small parameters $(p_0 + h\mathrm{p}) \ll |p_0|, \mathrm{p}$ $\quad M, a_L p_0, b_L \ll |p_0|, \mathrm{p}$

Density Matrix Equation

- From the kinetic equation we will get the density matrix equation:

$$\mathrm{i}\partial_t \mathcal{S}^+ + [\mathcal{H}, \mathcal{S}^+] - [\mathcal{N}, \mathcal{S}^H] \quad = \tfrac{1}{2}(\{\mathcal{G}^>, \mathcal{S}^<\} - \{\mathcal{G}^<, \mathcal{S}^>\}$$

- But we want an equation for on-shell distribution functions, while the above are matrices in spinor space that also exist off-shell. We Lorentz-decompose the propagators

$$\mathcal{S} = \sum_h \frac{1}{2} P_h \left(g_{0h} + \gamma^0 g_{1h} - \mathrm{i}\gamma^0\gamma^5 g_{2h} - \gamma^5 g_{3h} \right)$$

- Now we consider the constrained equation

$$\{\mathcal{H}, \mathcal{S}^+\} - \{\mathcal{N}, \mathcal{S}^H\} \quad = \tfrac{1}{2}([\mathcal{G}^>, \mathcal{S}^<] - [\mathcal{G}^<, \mathcal{S}^>])$$

- We multiply it with different combinations of gamma-matrices and take the trace to obtain relations between the Lorentz components

$$g_{1h} = \frac{1}{2p_0}\{g_{0h}, M\}, \quad g_{2h} = \frac{\mathrm{i}}{2p_0}[g_{3h}, M] \;, \quad g_{3h} = h\frac{p_0}{\mathrm{p}} g_{0h}$$

- Let's plug that into the propagator, and then insert the propagator back into the kinetic equation

Density Matrix Equation

- We find

$$
\begin{aligned}
\mathrm{tr}(P_s\mathcal{S}^+) &= g_{0s}^+ \\
\mathrm{tr}[\mathcal{H}, P_s\mathcal{S}^+] &= \left[\frac{1}{2p_0}M^2 + \frac{1}{2\mathrm{p}}\left(b_L(sp_0-\mathrm{p}) + a_L s(p_0^2-\mathrm{p}^2)\right), g_{0s}^+\right] \\
&\simeq \left[\frac{b_L}{2}(1-s\frac{p_0}{\mathrm{p}}) + \frac{1}{2p_0}M^2, g_{0s}^+\right] \\
\mathrm{tr}\{\mathcal{G}^{\gtrless}, P_s\mathcal{S}^{\lessgtr}\} &= \left\{\frac{1}{2p_0}M^2 + \frac{1}{2\mathrm{p}}\left(b_{\gtrless}(sp_0-\mathrm{p}) + a_L^{\gtrless} s(p_0^2-\mathrm{p}^2)\right), g_{0s}^{\lessgtr}\right\} \\
&\simeq \left\{-\frac{b_L^{\gtrless}}{2}(1-s\frac{p_0}{\mathrm{p}}), g_{0s}^{\lessgtr}\right\}
\end{aligned}
$$

- Comparing this to

$$
\begin{aligned}
\mathrm{tr}(\not{p}\Sigma) &= 2(b_L p_0 + a_L(p_0^2-\mathbf{p}^2)), \\
\mathrm{tr}(\not{p}P_h\Sigma) &= b_L(p_0-h\mathrm{p}) + a_L(p_0^2-\mathbf{p}^2)
\end{aligned}
$$

- We find

$$
\begin{aligned}
\mathrm{tr}[\mathcal{H}, P_s\mathcal{S}^+] &= \left[\frac{1}{2p_0}\left(\mathrm{tr}(\not{p}P_s\Sigma^{\mathrm{loc}}) + M^2\right), g_{0s}^+\right] \\
\mathrm{tr}\{\mathcal{G}^{\gtrless}, P_s\mathcal{S}^{\lessgtr}\} &= \left\{-\frac{1}{2p_0}\mathrm{tr}(\not{p}P_s\Sigma^{\gtrless}), g_{0s}^{\lessgtr}\right\}
\end{aligned}
$$

Now we just need to put this on-shell!

Density Matrix Equation

- The neutrino masses and matter potentials are kinematically completely negligible, they only matter in the (anti(commutators in the numerator of the propagator. Hence, we may use the pole structure of free spectral function in the relativistic limit for all propagators
- We split all quantities into an equilibrium part and a deviation

$$\mathcal{S}^{\cdots} + \bar{\mathcal{S}}^{\cdots} + \delta\mathcal{S}^{\cdots}$$

- Since the spectral function does not directly depend on the occupation numbers, we can neglect the deviation from equilibrium here. Since only two of the two-point functions are independent, this means

$$\delta\mathcal{S}^{>} = \delta\mathcal{S}^{<} = \delta\mathcal{S}^{+}$$

- The equilibrium pieces must fulfil the KMS relation
- Altogether this yields

$$\begin{aligned}
(g_{0h}^{+})_{\alpha\beta} &= 2\pi 2p_0\delta(p^2)\left[\left(\frac{1}{2} - f_F(p_0)\right)\delta_{\alpha\beta} - (\delta\rho(p))_{\alpha\beta}\right] \\
(g_{0h}^{-})_{\alpha\beta} &= 2\pi \mathrm{i} 2p_0\delta(p^2)\frac{1}{2}\delta_{\alpha\beta} \\
(g_{0h}^{<})_{\alpha\beta} &= 2\pi 2p_0\delta(p^2)[(-f_F(p_0))\delta_{\alpha\beta} - (\delta\rho(p))_{\alpha\beta}] \\
(g_{0h}^{>})_{\alpha\beta} &= 2\pi 2p_0\delta(p^2)[(1 - f_F(p_0))\delta_{\alpha\beta} - (\delta\rho(p))_{\alpha\beta}]
\end{aligned}$$

Density Matrix Equation

- Plugging this back into the kinetic equation …

$$i\partial_t \mathcal{S}^+ + [\mathcal{H}, \mathcal{S}^+] - [\mathcal{N}, \mathcal{S}^H] = \frac{1}{2}(\{\mathcal{G}^>, \mathcal{S}^<\} - \{\mathcal{G}^<, \mathcal{S}^>\})$$

- …and integrating over p0 $\rho = \int \frac{dp_0}{2\pi} \mathrm{tr}\mathcal{S}^+.$ gives

$$\partial_t \rho = -i[\mathbb{H}, \rho] + \mathcal{I}[\rho]$$

$$\mathcal{I}[\rho] = \frac{1}{2}(\{\varGamma^>, -\rho\} + \{\varGamma^<, 1-\rho\}) = \frac{1}{2}\Big((1-\rho)\varGamma^< - \rho\varGamma^>\Big) + \mathrm{h.c.}$$

- with

$$\mathbb{H} = \frac{1}{2p_0}\left(\mathrm{tr}(\not{p} P_s \varSigma^{\mathrm{loc}}) + M^2\right)\Big|_{p_0=\varOmega_{\mathbf{p}}} \qquad \varGamma^{\gtrless} = \frac{\mp 1}{2p_0}\mathrm{tr}(\not{p}\varSigma^{\gtrless})\Big|_{p_0=\varOmega_{\mathbf{p}}}$$

- The expansion of the universe is finally included by interpreting this equation as one in conformal time and comoving coordinates

Connecting to Boltzmann

- We found for the gain and loss rates

$$\Gamma^{\gtrless} = \frac{\mp 1}{2p_0} \mathrm{tr}(\not{p}\Sigma^{\gtrless})|_{p_0=\Omega_{\mathbf{p}}}$$

- Can we somehow connect that to the collision term in the Boltzmann equation?

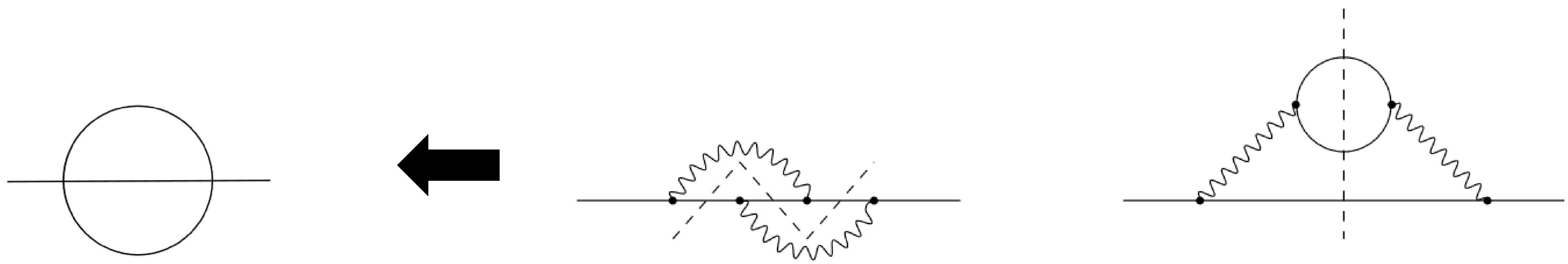
- We consider the total damping rate

$$\Gamma_{\mathbf{p}} = \Gamma_{\mathbf{p}}^{>} + \Gamma_{\mathbf{p}}^{<} = \frac{1}{p_0} \mathrm{Im\,tr}(\not{p}\Sigma^{R}(p)\Big|_{p_0=\Omega_{\mathbf{p}}}$$

- And a generic interaction like

$$\mathcal{L}_{\mathrm{Fermi}} = 2\sqrt{2}G_F(\bar{\nu}\gamma^{\mu}P_L\Psi_3)\Big(\bar{\Psi}_1\gamma_{\mu}(c_V' + c_A'\gamma^5)\Psi_2\Big) + \mathrm{h.c.},$$

- There can be two types of fermion flow in “setting sun” diagrams, depending on where they come from



Connecting to Boltzmann

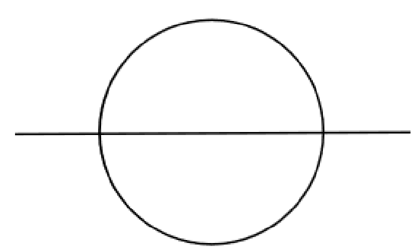
- They read

$$\mathrm{Im}\,\hat{\not{\Sigma}}^{R,1}(q) = 4G_F^2\Big[\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2+p_3-q)$$

$$\gamma^\mu \mathrm{Tr}\Big[\gamma_\mu(a+b\gamma^5)S^>(p_1)\gamma_\nu(a+b\gamma^5)S^<(-p_2)\Big]S^>(p_3)\gamma^\nu\Big) - S^> \leftrightarrow S^<\Big]$$

$$\mathrm{Im}\,\hat{\Sigma}^{R,2}(q) = 4G_F^2\Big[\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2+p_3-q)$$

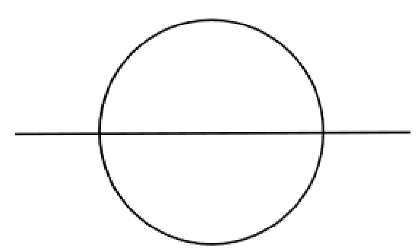
$$\Big(\gamma^\mu S^>(p_3)\gamma^\nu(a+b\gamma^5)S^<(-p_2)\gamma_\mu(c+d\gamma^5)S^>(p_1)\gamma_\nu\Big) - S^> \leftrightarrow S^<\Big].$$



Connecting to Boltzmann

- Inerting the propagators gives

$$\begin{aligned}\mathrm{Im}\,\hat{\not{\Sigma}}^{R,1}(q) =& 4G_F^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2+p_3-q)\Big[(1-f_1)(1-f_2)(1-f_3) \\ &+ f_1f_2f_3\Big]\gamma^\mu \mathrm{Tr}\Big[\gamma_\mu(a+b\gamma^5)\rho(p_1)\gamma_\nu(a+b\gamma^5)\rho(-p_2)\Big]\rho(p_3)\gamma^\nu \\ \mathrm{Im}\,\hat{\Sigma}^{R,2}(q) =& 4G_F^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2+p_3-q)\Big[(1-f_1)(1-f_2)(1-f_3) \\ &+ f_1f_2f_3\Big]\Big(\gamma^\mu\rho(p_3)\gamma^\nu(a+b\gamma^5)\rho(-p_2)\gamma_\mu(c+d\gamma^5)\rho(p_1)\gamma_\nu\Big).\end{aligned}$$



Connecting to Boltzmann

- And yes, this exactly gives back the Boltzmann integral!

$$\mathrm{Tr}\left[\not{K}\,\mathrm{Im}\,\hat{\Sigma}^{\mathrm{ret},s}_{L/R}(q)\right] = (2\pi)^4 \int \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_1} \int \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_2} \int \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2E_3} \times \qquad (
$$

$$\begin{aligned}
\times\Big[&\delta^4(p_1+p_2+p_3-q)|\mathcal{M}^s_K(m_1,-m_2,m_3)|^2[(1-f_1)(1-f_2)(1-f_3)+f_1f_2f_3] \\
&+\delta^4(p_1+p_2-p_3-q)|\mathcal{M}^s_K(m_1,-m_2,-m_3)|^2[(1-f_1)(1-f_2)f_3+f_1f_2(1-f_3)] \\
&+\delta^4(p_1+p_3-p_2-q)|\mathcal{M}^s_K(m_1,m_2,m_3)|^2[(1-f_1)f_2(1-f_3)+f_1(1-f_2)f_3] \\
&+\delta^4(p_2+p_3-p_1-q)|\mathcal{M}^s_K(-m_1,-m_2,m_3)|^2[f_1(1-f_2)(1-f_3)+(1-f_1)f_2f_3] \\
&+\delta^4(p_3-p_1-p_2-q)|\mathcal{M}^s_K(-m_1,m_2,m_3)|^2[f_1f_2(1-f_3)+(1-f_1)(1-f_2)f_3] \\
&+\delta^4(p_2-p_1-p_3-q)|\mathcal{M}^s_K(-m_1,-m_2,-m_3)|^2[f_1(1-f_2)f_3+(1-f_1)f_2(1-f_3)] \\
&+\delta^4(p_1-p_2-p_3-q)|\mathcal{M}^s_K(m_1,m_2,-m_3)|^2[(1-f_1)f_2f_3+f_1(1-f_2)(1-f_3)] \\
&+\delta^4(-p_1-p_2-p_3-q)|\mathcal{M}^s_K(-m_1,m_2,-m_3)|^2[f_1f_2f_3+(1-f_1)(1-f_2)(1-f_3)]\Big]
\end{aligned}$$

Kinematics | Matrix element | Quantum statistics

$$|\mathcal{M}^s_K(m_1,m_2,m_3;p_1,p_2,p_3,q)|^2 \equiv |\mathcal{M}^s_K(m_1,m_2,m_3)|^2 = 4G_F^2\mathcal{T}^s_K(m_1,m_2,m_3)$$

N_{eff} in the Standard Model

- N_{eff} is the "effective number of neutrino species"; parameterises expansion rate during BBN and CMB decoupling
- In the SM it is defined as

$$\left.\frac{\rho_\nu}{\rho_\gamma}\right|_{T/m_e\to 0} \equiv \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}^{\rm SM}$$

- $N_{eff} = 3$ in the SM if
  - i) primordial plasma is ideal gas,
  - ii) neutrinos decouple instantaneously,
  - iii) neutrinos decoupled at temperature $T >> m_e$

  None of this is really true, leading to deviations from $N_{eff} = 3$

- BSM phenomena also lead to deviations from $N_{eff} = 3$ (extra light particles, modified expansion history, non-standard neutrino interactions…), making N_{eff} a powerful probe of BSM physics
- CMB S4 can potentially measure N_{eff} with sub-percent accuracy

- Computed the current state-of-the art value in the SM

$$N_{\rm eff}^{\rm SM} = 3.0440 \pm 0.0002$$

Froustey/Pitrou et al 2008.01074
Akita/Yamaguchi al 2005.07047
Bennet et al 2012.02726

(used by PDG, in CAMB and CLASS codes, major collaborations like DES, DESI…)

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  None of this is really true, leading to deviations from $N_{eff} = 3$

$$\rho_\gamma + \rho_\nu + [\text{new physics}] \equiv \rho_\gamma + N_{\rm eff}\,\rho_\nu = \frac{\pi^2}{15} T_\gamma^4 \left[1 + N_{\rm eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]$$

- CMB S4 can potentially measure N_{eff} with sub-percent accuracy

- Computed the current state-of-the art value in the SM

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QED Equation of State

QED equation of state can be computed from partition function Z

In practice lnZ is expanded in powers of e

$$\ln Z = \ln Z^{(0)} + \ln Z^{(2)} + \ln Z^{(3)} + \cdots$$

From this, contributions to energy, pressure and entropy care computed

$$P^{(n)} = \frac{T}{V} \ln Z^{(n)},$$

$$\rho^{(n)} = \frac{T^2}{V} \frac{\partial \ln Z^{(n)}}{\partial T} = -P^{(n)} + T \frac{\partial P^{(n)}}{\partial T},$$

$$s^{(n)} = \frac{1}{V} \frac{\partial \left[T \ln Z^{(n)} \right]}{\partial T} = \frac{\rho^{(n)} + P^{(n)}}{T},$$

QED Equation of State

At zeroth order one finds ideal gas

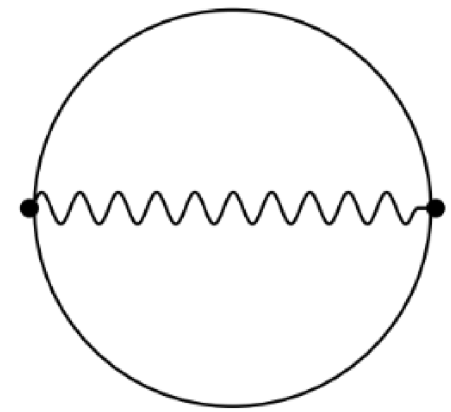
$$P^{(0)} = \frac{T}{\pi^2} \int_0^\infty \mathrm{d}p\, p^2 \;\ln\left[\frac{(1+\mathrm{e}^{-E_e/T})^2}{(1-\mathrm{e}^{-E_\gamma/T})}\right],$$

$$\rho^{(0)} = \frac{1}{\pi^2} \int_0^\infty \mathrm{d}p\, p^2 \left[\frac{2E_e}{e^{E_e/T}+1} + \frac{E_\gamma}{e^{E_\gamma/T}-1}\right],$$

QED Equation of State O(e²)

The first correction comes from

$$\ln Z^{(2)} = -\tfrac{1}{2}$$



$$P^{(2)} = \frac{T}{V}\ln Z^{(2)} = -\frac{e^2T^2}{12\pi^2}\int_0^\infty \mathrm{d}p\, \frac{p^2}{E_p} n_D - \frac{e^2}{8\pi^4}\left(\int_0^\infty \mathrm{d}p\, \frac{p^2}{E_p} n_D\right)^2 + \frac{e^2 m_e^2}{16\pi^4}\iint_0^\infty \mathrm{d}p\, \mathrm{d}\tilde{p}\, \frac{p\tilde{p}}{E_p E_{\tilde{p}}}\, \ln\left|\frac{p+\tilde{p}}{p-\tilde{p}}\right|\, n_D \tilde{n}_D,$$

Usually the log-dependent term is neglected

QED Equation of State O(e²)[no log]

Neglecting the log term yields

$$P^{(2)} = \frac{T}{V} \ln Z^{(2)} = -\frac{e^2 T^2}{12\pi^2} \int_0^\infty \mathrm{d}p \, \frac{p^2}{E_p} n_D - \frac{e^2}{8\pi^4} \left(\int_0^\infty \mathrm{d}p \, \frac{p^2}{E_p} n_D \right)^2$$

$$\rho^{(2)\not{\ln}} = -\frac{e^2 T^2}{12\pi^2} \int_0^\infty \mathrm{d}p \frac{p^2}{E_p} \left(n_D + T \partial_T n_D \right) + \frac{e^2}{8\pi^4} \left(\int_0^\infty \mathrm{d}p \frac{p^2}{E_p} n_D \right)^2 - \frac{e^2}{4\pi^4} \left(\int_0^\infty \mathrm{d}p \frac{p^2}{E_p} \, n_D \right) \left(\int_0^\infty \mathrm{d}p \frac{p^2}{E_p} \, T \partial_T n_D \right),$$

QED Equation of State O(e²)

Adding the log term yields

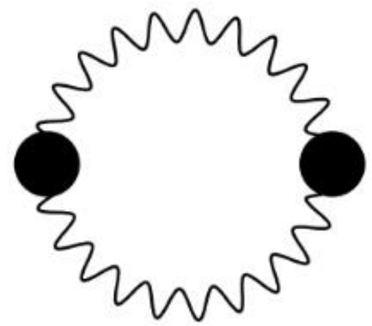
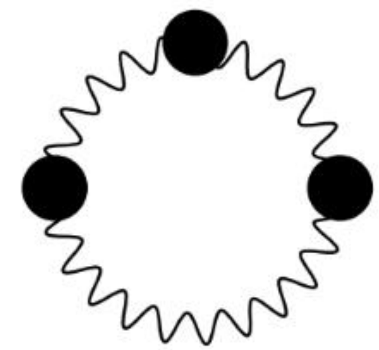
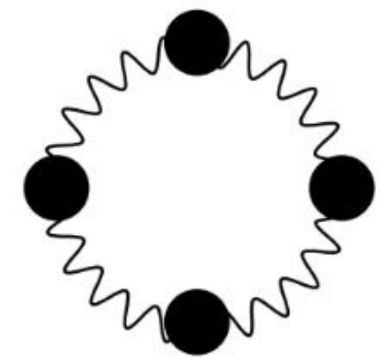
$$P^{(2)} = \frac{T}{V}\ln Z^{(2)} = -\frac{e^2T^2}{12\pi^2}\int_0^\infty \mathrm{d}p\,\frac{p^2}{E_p}n_D - \frac{e^2}{8\pi^4}\left(\int_0^\infty \mathrm{d}p\,\frac{p^2}{E_p}n_D\right)^2 + \frac{e^2m_e^2}{16\pi^4}\iint_0^\infty \mathrm{d}p\,\mathrm{d}\tilde{p}\,\frac{p\tilde{p}}{E_pE_{\tilde{p}}}\,\ln\left|\frac{p+\tilde{p}}{p-\tilde{p}}\right|\,n_D\tilde{n}_D,$$

$$\rho^{(2)\not{\ln}} = -\frac{e^2T^2}{12\pi^2}\int_0^\infty \mathrm{d}p\frac{p^2}{E_p}\left(n_D + T\partial_T n_D\right) + \frac{e^2}{8\pi^4}\left(\int_0^\infty \mathrm{d}p\frac{p^2}{E_p}n_D\right)^2 - \frac{e^2}{4\pi^4}\left(\int_0^\infty \mathrm{d}p\frac{p^2}{E_p}\,n_D\right)\left(\int_0^\infty \mathrm{d}p\frac{p^2}{E_p}\,T\partial_T n_D\right),$$

$$\rho^{(2)\,\ln} = \frac{e^2m_e^2}{16\pi^4}\iint_0^\infty \mathrm{d}p\,\mathrm{d}\tilde{p}\,\frac{p\tilde{p}}{E_pE_{\tilde{p}}}\ln\left|\frac{p+\tilde{p}}{p-\tilde{p}}\right|\,n_D\left(2T\partial_T\tilde{n}_D - \tilde{n}_D\right)$$

QED Equation of State O(e³)

The next correction comes from

$$\ln Z^{(3)} = \frac{1}{2}\Bigg[\ \frac{1}{2}$$

$$-\ \frac{1}{3}$$

$$+\ \frac{1}{4}$$

$$+\ \cdots\ \Bigg]$$

It reads

$$P^{(3)} = \frac{T}{V}\ln Z^{(3)} = \frac{e^3 T}{12\pi^4} I^{3/2}(T) \qquad \rho^{(3)} = \frac{e^3 T^2}{8\pi^4} I^{1/2}\partial_T I$$

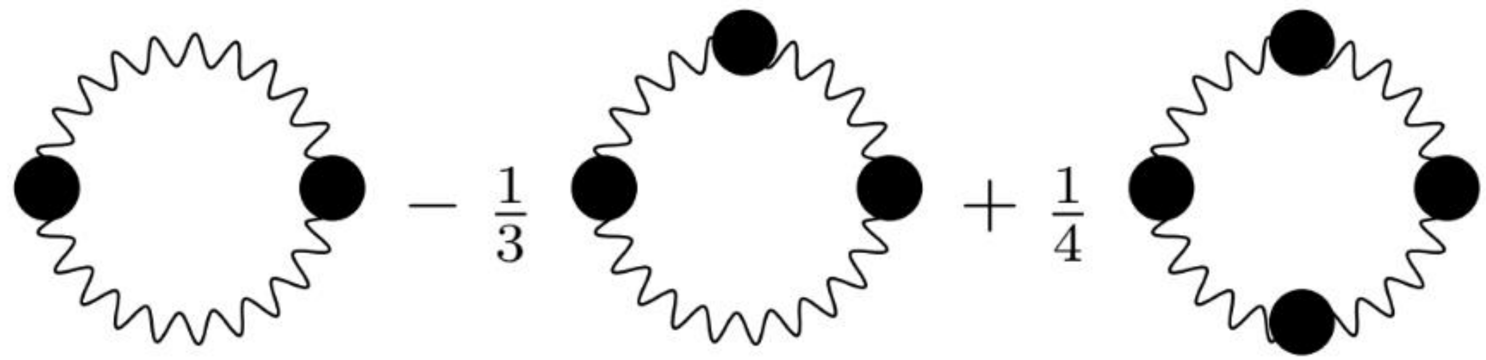
with

$$I(T) = \int_0^\infty \mathrm{d}p \left(\frac{p^2 + E_p^2}{E_p}\right) n_D$$

QED Equation of State O(e³)

The next correction comes from

$$\ln Z^{(3)} = \frac{1}{2}\left[\frac{1}{2}\,(\ldots) - \frac{1}{3}\,(\ldots) + \frac{1}{4}\,(\ldots) + \cdots \right]$$



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with

$$I(T) = \int_0^\infty \mathrm{d}p \left(\frac{p^2 + E_p^2}{E_p}\right) n_D$$

- This correction was previously neglected
- It turns out to be important

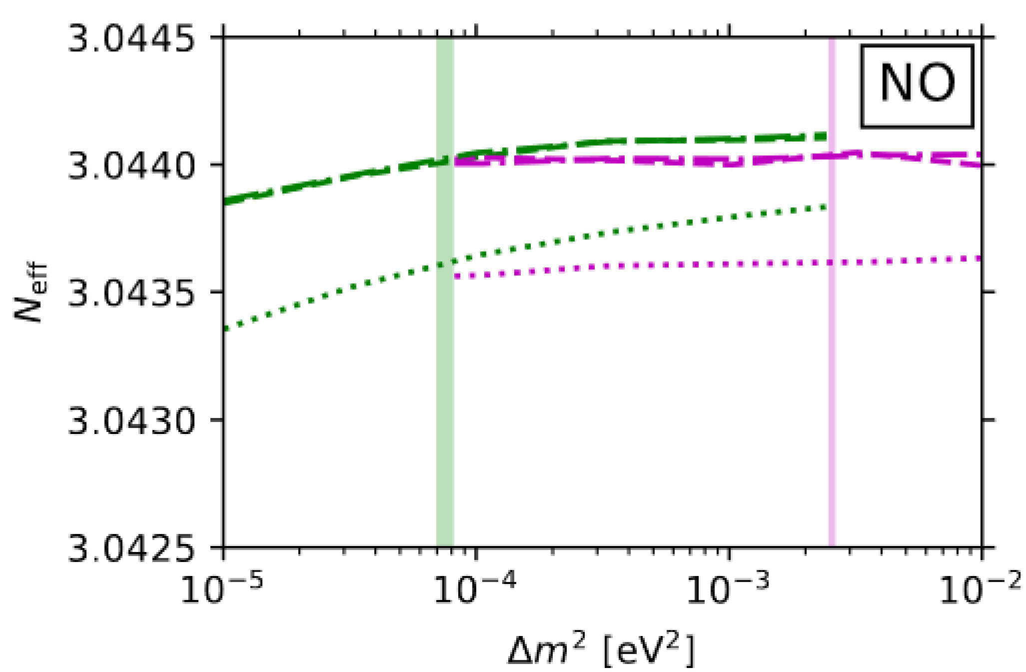
Impact of NLO QED Corrections

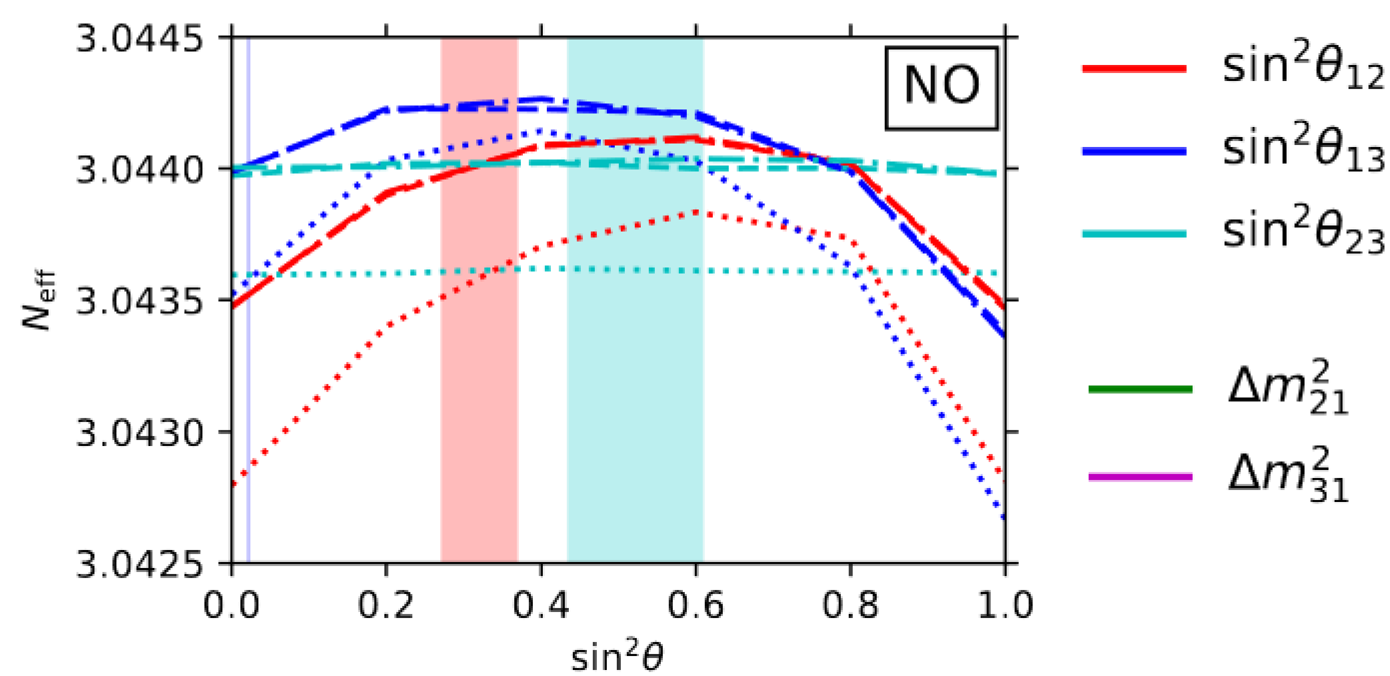
- *Neff* in the SM is found by solving

$$(\mathrm{d}/\mathrm{d}t)\rho_{\mathrm{tot}} + 3H(\rho_{\mathrm{tot}} + P_{\mathrm{tot}}) = 0, \qquad \partial_t \varrho - pH\partial_p = -\mathrm{i}[\mathbb{H}, \varrho] + \mathcal{I}[\varrho],$$

- O(e³) correction to equation of state is more important than neutrino oscillations!

Standard-model corrections to $N_{\mathrm{eff}}^{\mathrm{SM}}$	Leading-digit contribution
m_e/T_d correction	$+0.04$
$\mathcal{O}(e^2)$ FTQED correction to the QED EoS	$+0.01$
Non-instantaneous decoupling+spectral distortion	-0.005
$\mathcal{O}(e^3)$ FTQED correction to the QED EoS	-0.001
Flavour oscillations	$+0.0005$
Type (a) FTQED corrections to the weak rates	$\lesssim 10^{-4}$





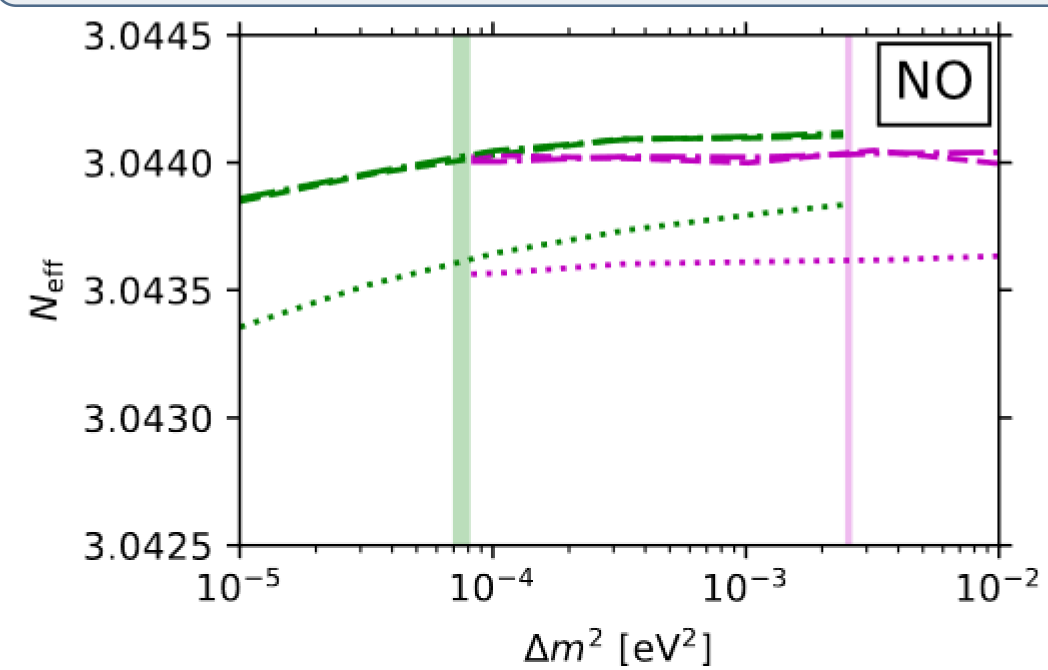
Impact of NLO QED Corrections

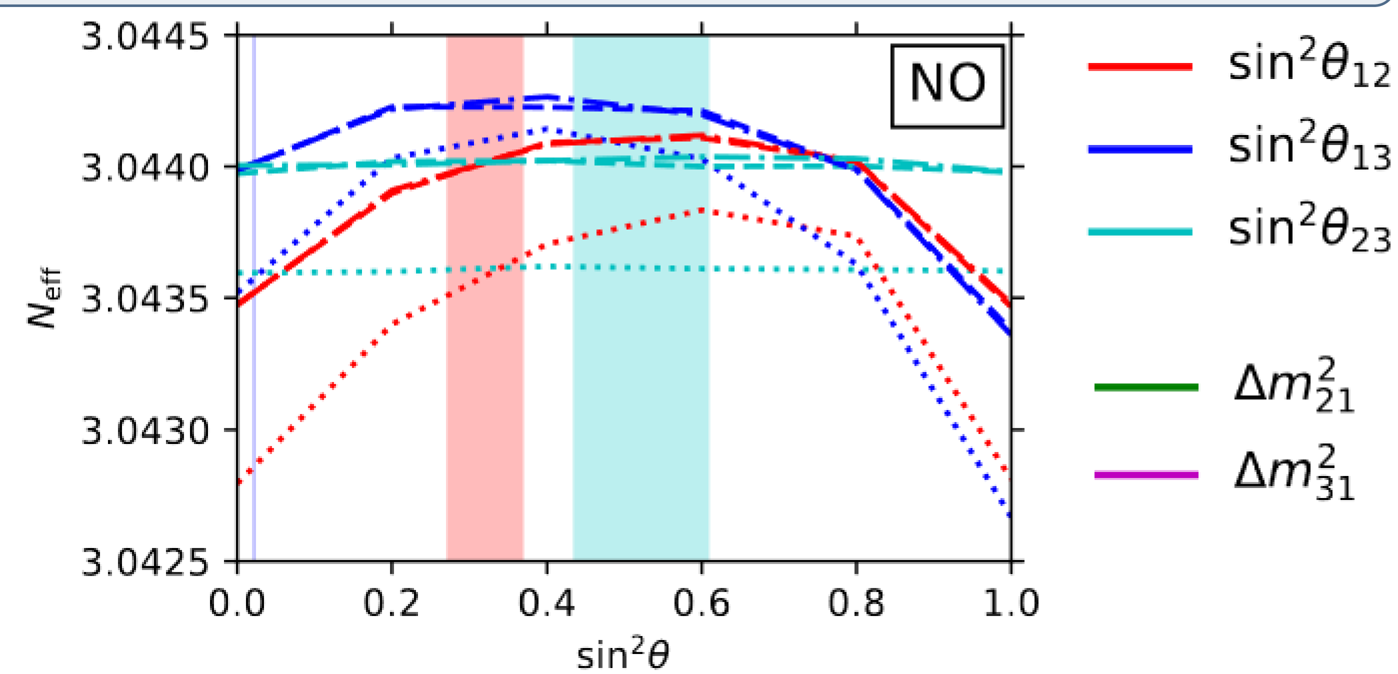
- *Neff* in the SM is found by solving

$$(\mathrm{d}/\mathrm{d}t)\rho_{\rm tot} + 3H(\rho_{\rm tot} + P_{\rm tot}) = 0, \qquad \partial_t \varrho - pH\partial_p = -\mathrm{i}[\mathbb{H}, \varrho] + \mathcal{I}[\varrho],$$

- First inclusion of NLO correction to equation of state at $O(e^3)$
  → is more important than neutrino oscillations!
- Estimate of NLO correction to collision term
  → is negligible (in contrast to claims in the literature).
- First inclusion of experimental error in ν-oscillation parameters
  → is subdominant

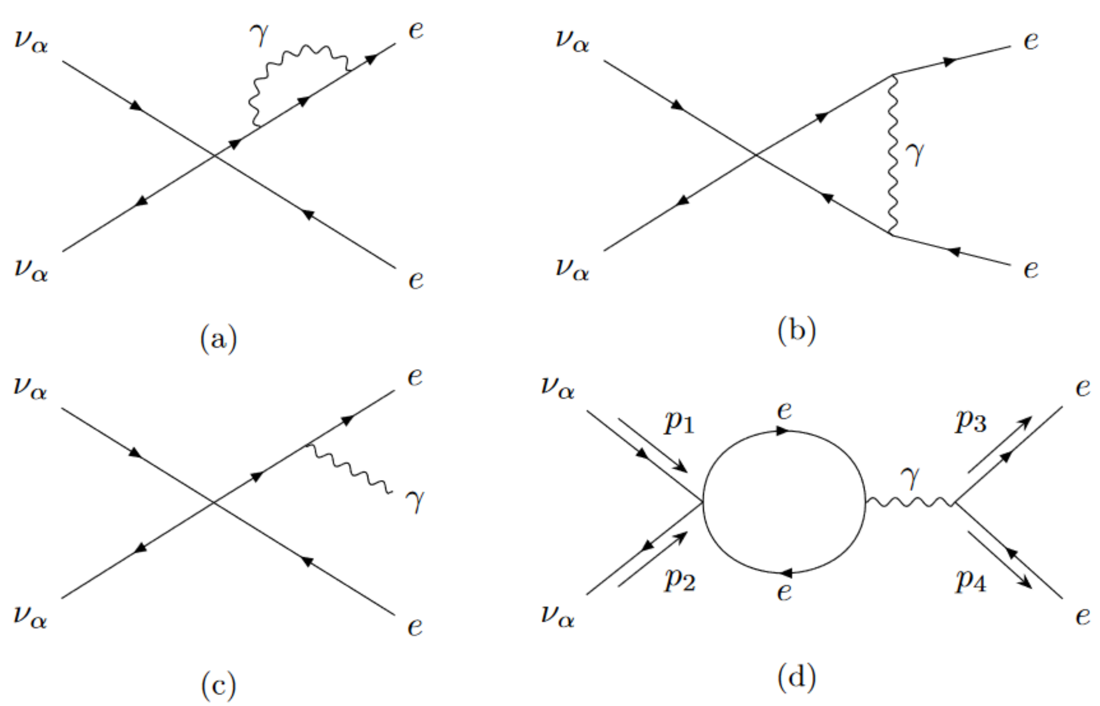
Bennet et al 1911.04504
Bennet et al 2012.02726
Drewes et al 2402.18481

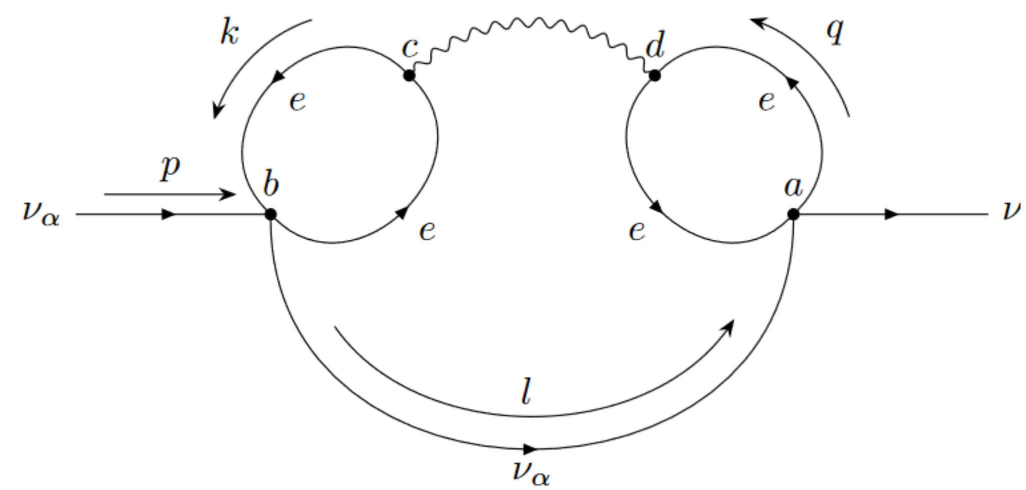




QED Corrections to Collision Term

The following corrections to the collision term were not included



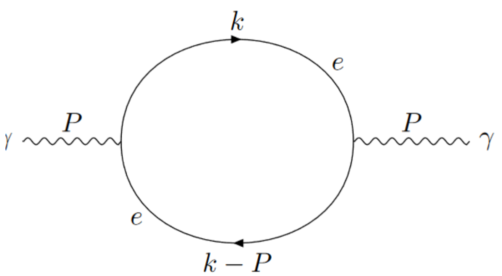


There is controversy about their impact

Cielo et al 2306.05460 , Jackson/laine 2312.07015, Drewes et al 2402.18481

Diagram (d) is IR divergent in the t-channel, requires usage of resumed finite temperature photon propagator

Resummed Photon Propagator



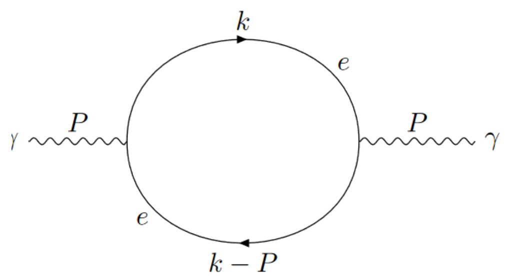
$$\Pi_{ab}^{\mu\nu}(P) = (-1)^{a+b} ie^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[iS_e^{ab}(k)\gamma^\mu iS_e^{ba}(k-P)\gamma^\nu \right]$$

$$\begin{aligned}\mathrm{Re}\,\Pi^T_{11,T\neq 0} = \frac{\alpha_{\mathrm{em}}}{\pi|\mathbf{P}|^3}\int_{m_e}^{\infty} \mathrm{d}\omega \, \Big[2\ell_2(\omega,P)P_0P^2\omega - 4|\mathbf{k}||\mathbf{P}|(P_0^2+|\mathbf{P}|^2) \\ +\ell_1(\omega,P)(|\mathbf{P}|^2P^2 + 2P_0^2\omega^2 - 2|\mathbf{k}|^2|\mathbf{P}|^2 + \frac{1}{2}P^4)\Big] f_F(\omega) \\ \overset{\mathrm{HTL}}{\approx} -\frac{3}{2}m_\gamma^2 \frac{P_0^2}{|\mathbf{P}|^2}\left[1-\left(1-\frac{|\mathbf{P}|^2}{P_0^2}\right)\frac{P_0}{2|\mathbf{P}|}\ln\left|\frac{P_0+|\mathbf{P}|}{P_0-|\mathbf{P}|}\right|\right] \longrightarrow 0, \quad \text{for } P_0 = 0\end{aligned}$$

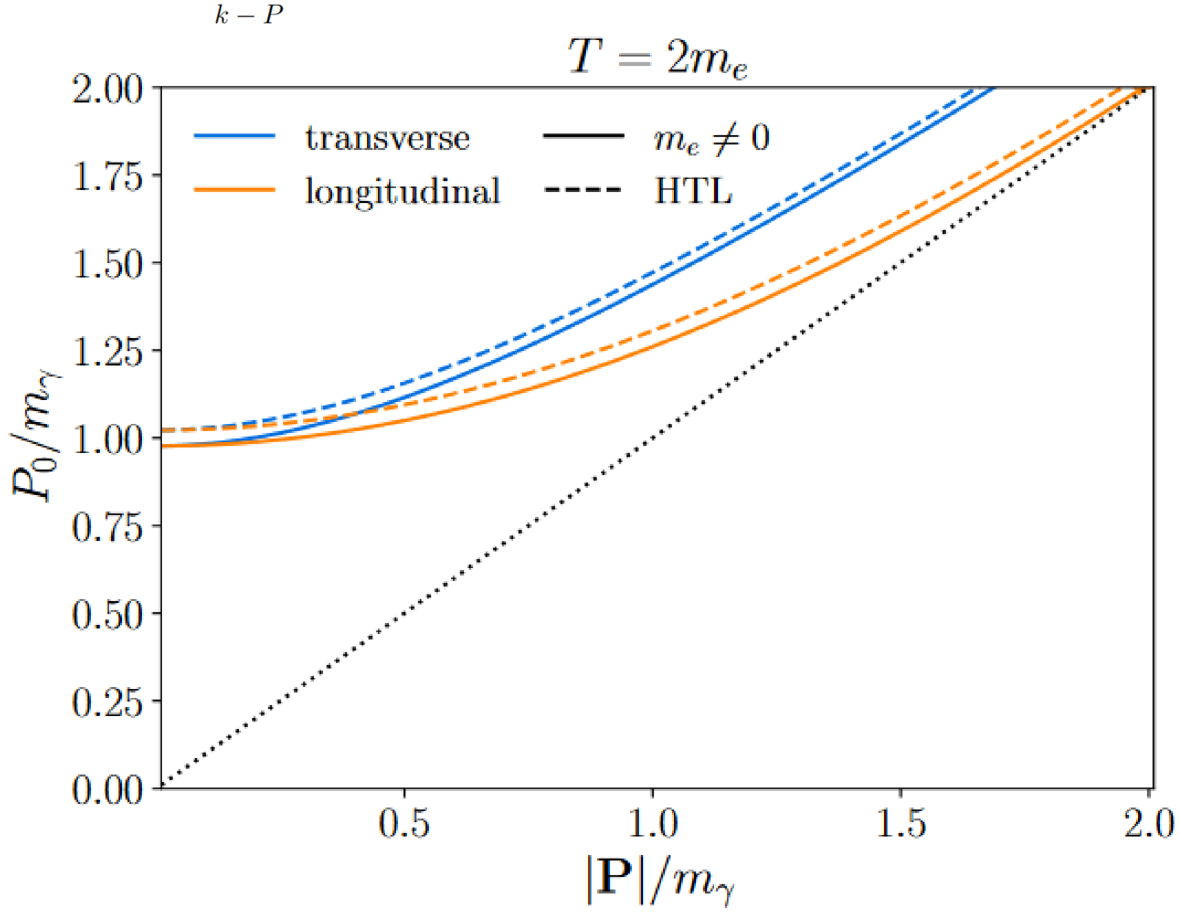
$$\begin{aligned}\mathrm{Re}\,\Pi^L_{11,T\neq 0} = \frac{\alpha_{\mathrm{em}}P^2}{\pi|\mathbf{P}|^3}\int_{m_e}^{\infty} \mathrm{d}\omega \, \left[8|\mathbf{k}||\mathbf{P}| - \ell_1(\omega,P)(P^2+4\omega^2) - 4\ell_2(\omega,P)P_0\omega\right] f_F(\omega) \\ \overset{\mathrm{HTL}}{\approx} -3m_\gamma^2\left(1-\frac{P_0^2}{|\mathbf{P}|^2}\right)\left[1-\frac{P_0}{2|\mathbf{P}|}\ln\left|\frac{P_0+|\mathbf{P}|}{P_0-|\mathbf{P}|}\right|\right] \longrightarrow -3m_\gamma^2 \quad \text{for } P_0 = 0,\end{aligned}$$

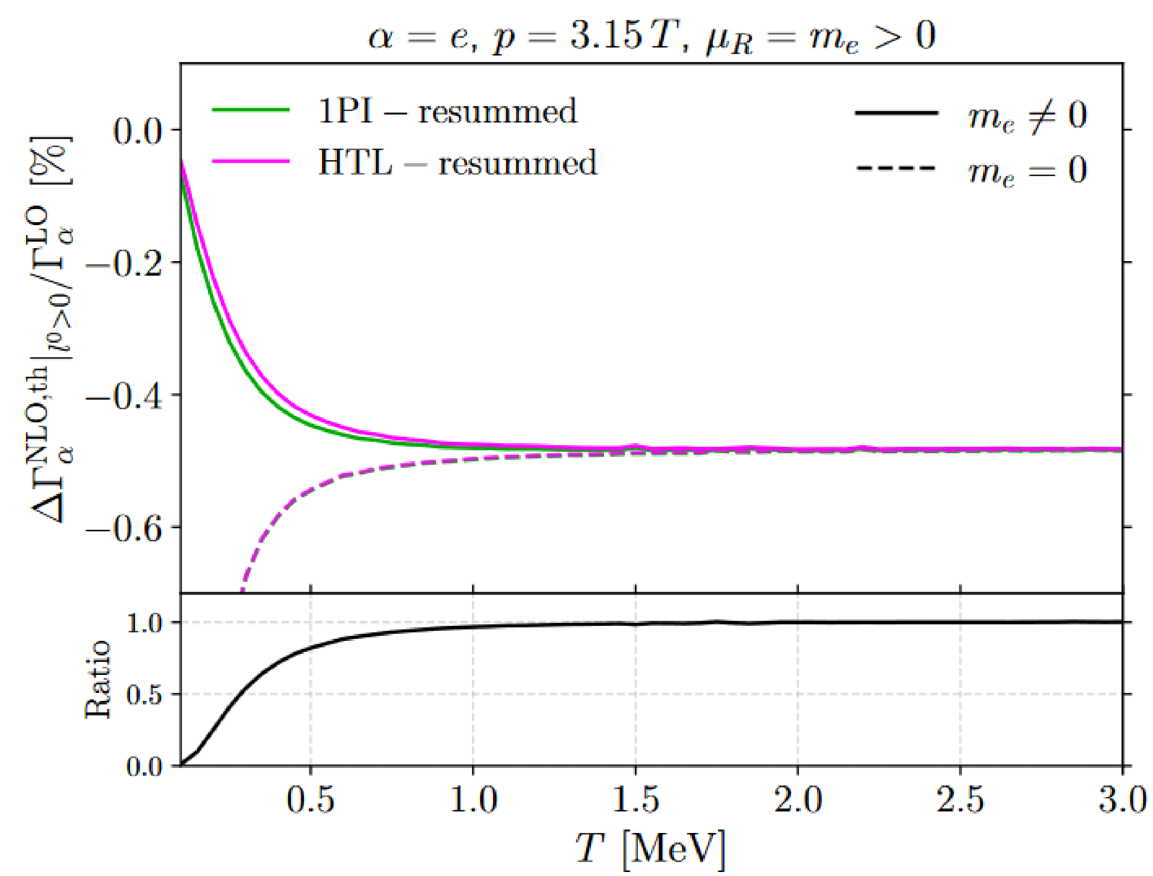
Resummation introduces momentum-dependent photon mass and longitudinal photon mode

Resummed Photon Propagator

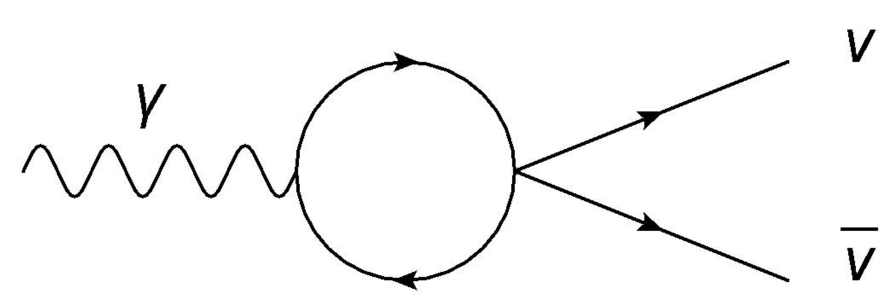


$$\Pi^{\mu\nu}_{ab}(P) = (-1)^{a+b} i e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[i S_e^{ab}(k) \gamma^\mu i S_e^{ba}(k-P) \gamma^\nu \right]$$

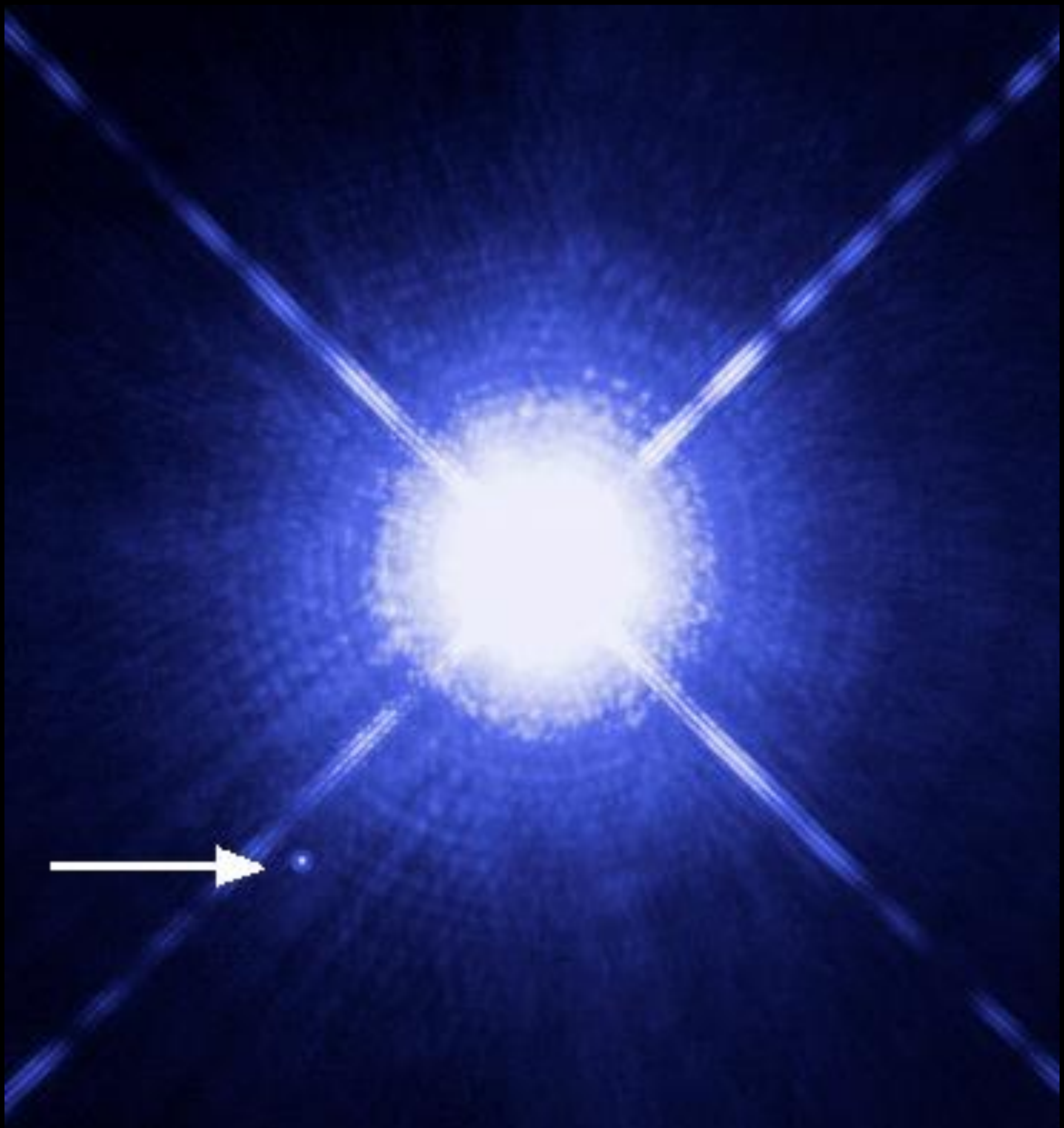




Impact on Neff is subdominant, but it gives rise to well-known plasmon process that is relevant for White Dwarf coolings

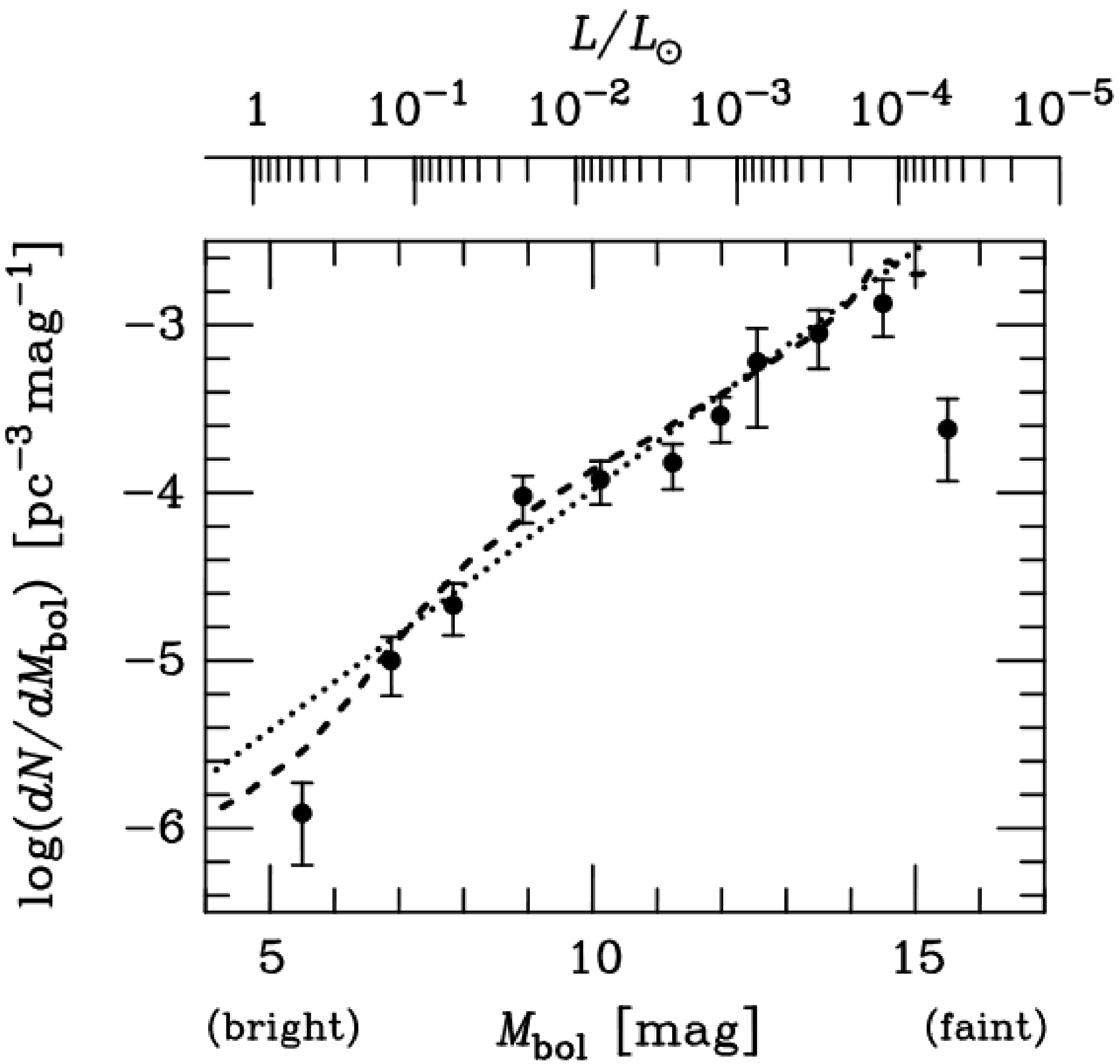


White Dwarf Cooling

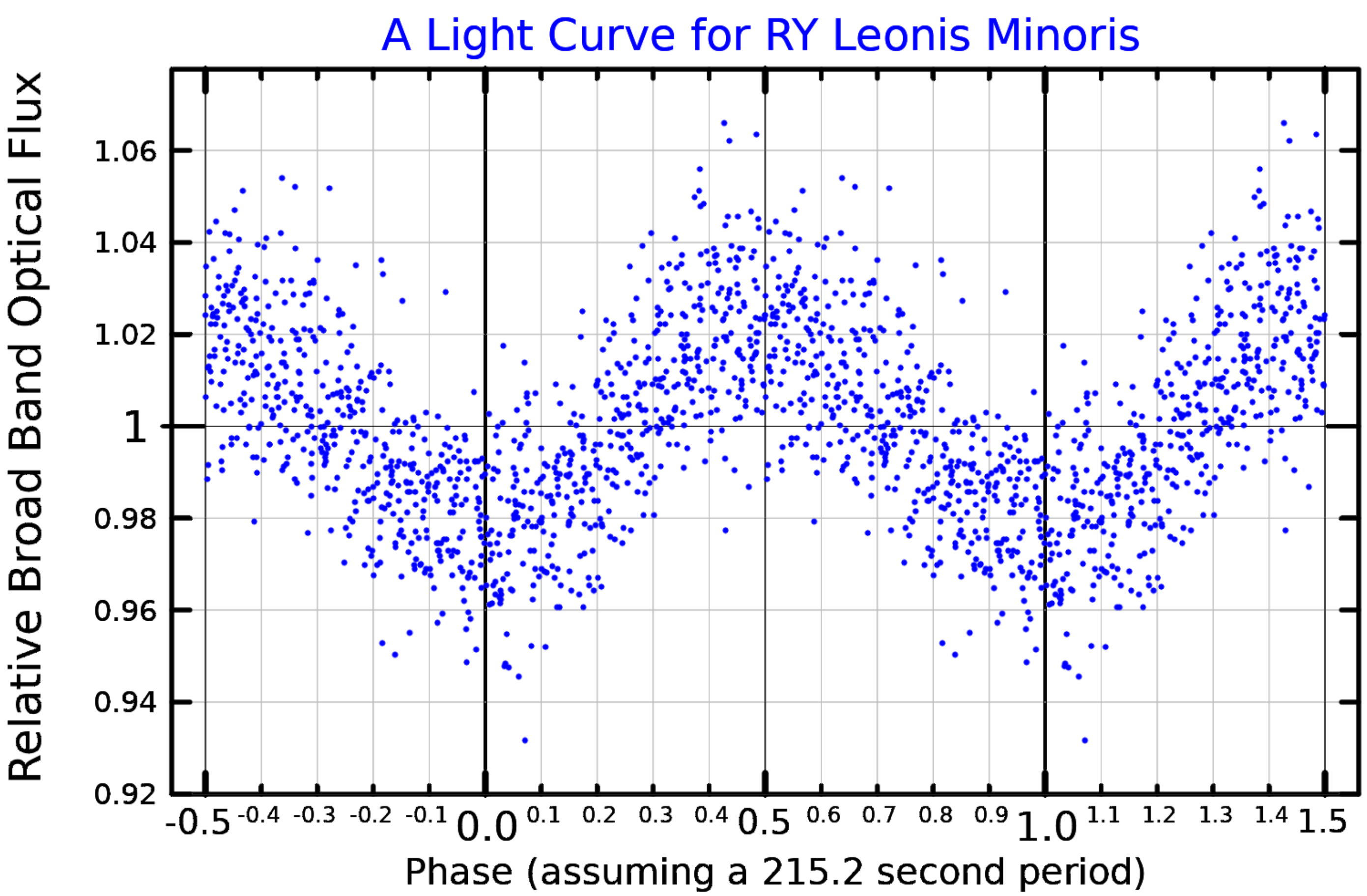


Observing WD Cooling

White Dwarf Luminosity Function (WDLF): Population



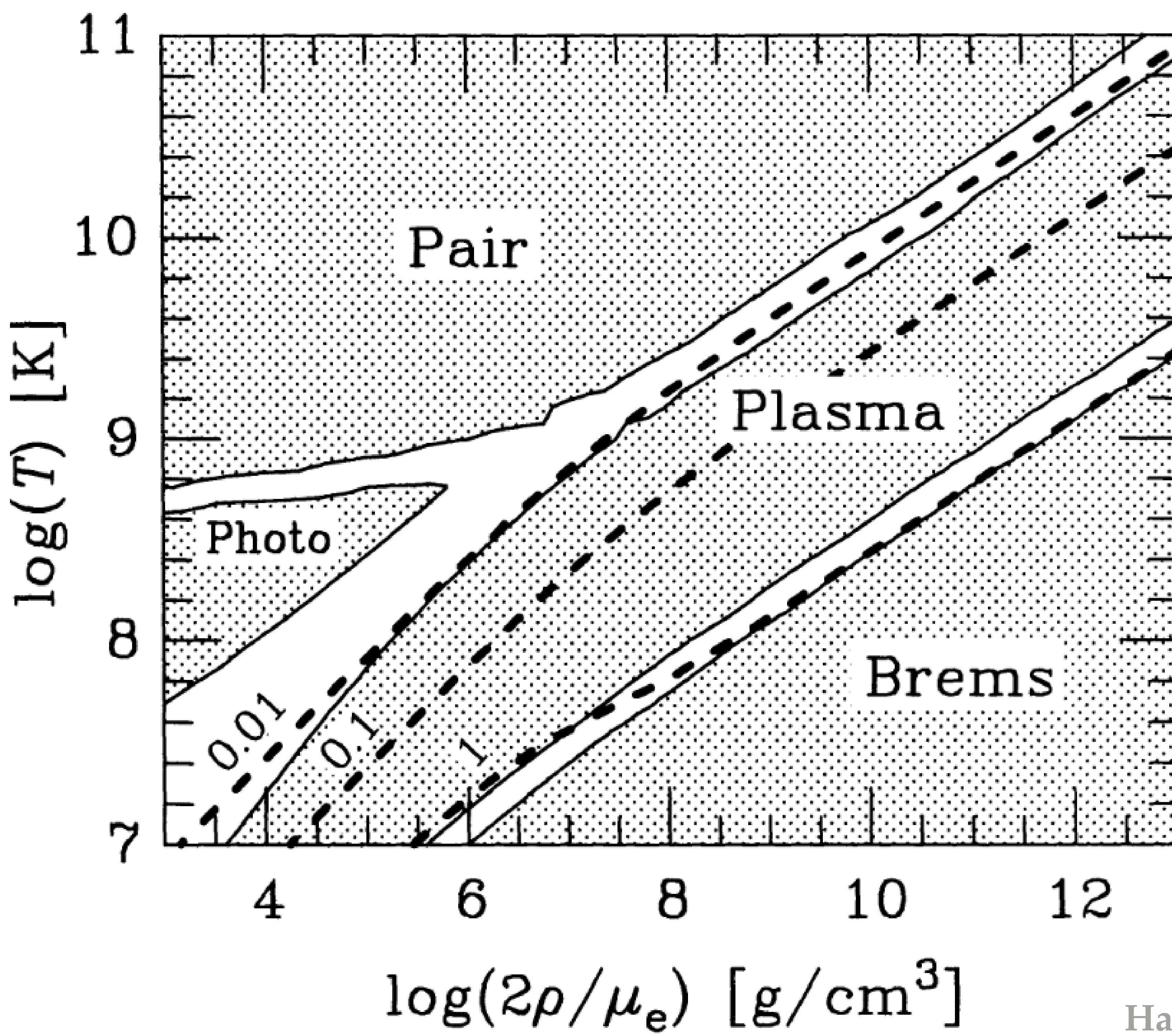
Pulsating WDs: Individual stars



G117-B15A

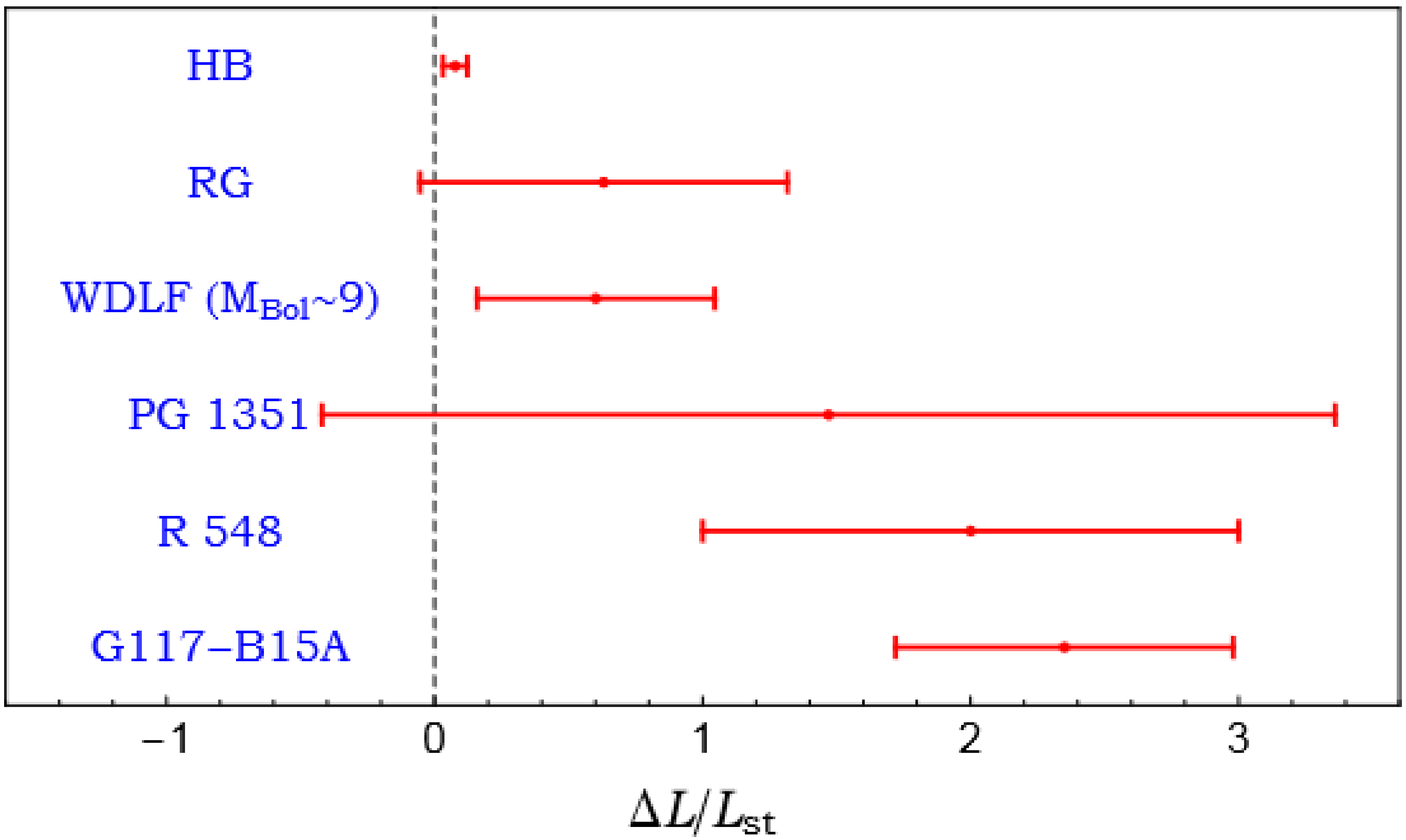
$$T = 1.2 \times 10^7 \text{ K} \quad Y_e\rho \simeq 10^6 \text{ g/cm}^3$$

Theory of WD Cooling



Haft/Raffelt/Weiss 1994

Cooling Anomaly



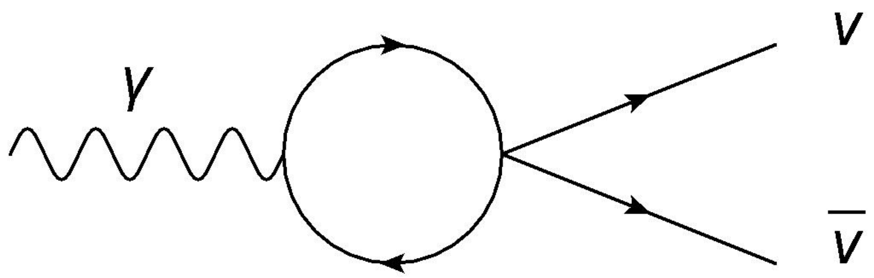
- Some WDs appear to be cooling too fast….
- Do they emit LLPs (axions, ALPs, …)?

Giannotti/Irastorza/Redondo/Ringwald 1512.08108

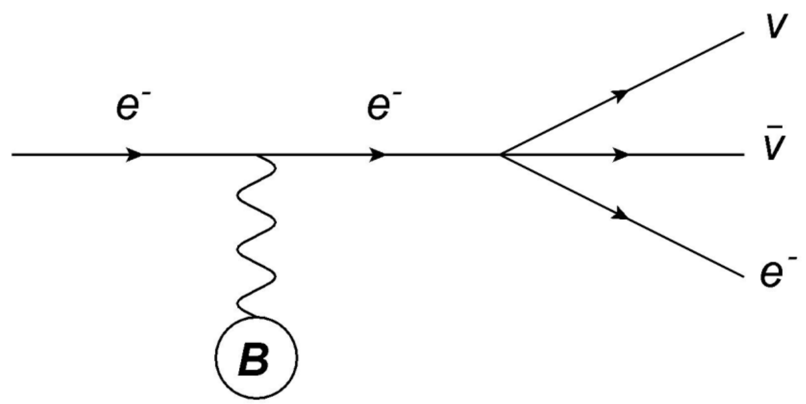
Impact of B-fields

Can internal magnetic fields explain this within the SM?

- Modify plasma processes $\gamma \rightarrow \nu\nu$



- Enable synchroton radiation $e \rightarrow e\nu\nu$



- Heating through Ohmic decay

Plasma Processes

- $\gamma \to \nu\nu$ possible in dense medium due to modified photon and plasmon dispersion relations, roughly characterised by the plasma frequency

cf. e.g. Braaten/Segel 9302213

$$\omega_p^2 = \frac{4\pi\alpha\, n_e}{m_e}\left[1 + \frac{1}{m_e^2}(3\pi^2 n_e)^{2/3}\right]^{-1/2} \simeq \left(20\,\mathrm{keV}\,\rho_6^{1/2}\right)^2$$

- Refractive index ("thermal mass") is determined by electron density, relevant scale is the frequency

cf. e.g. Kennet/Melrose astro-ph/9901156

$$\omega_B = \frac{eB}{m_e} = m_e \frac{B}{B_c} \simeq 11.5\, B_{12}\,\mathrm{keV} \qquad B_c = m_e^2/e = 4.41 \times 10^{13}\,\mathrm{G}$$

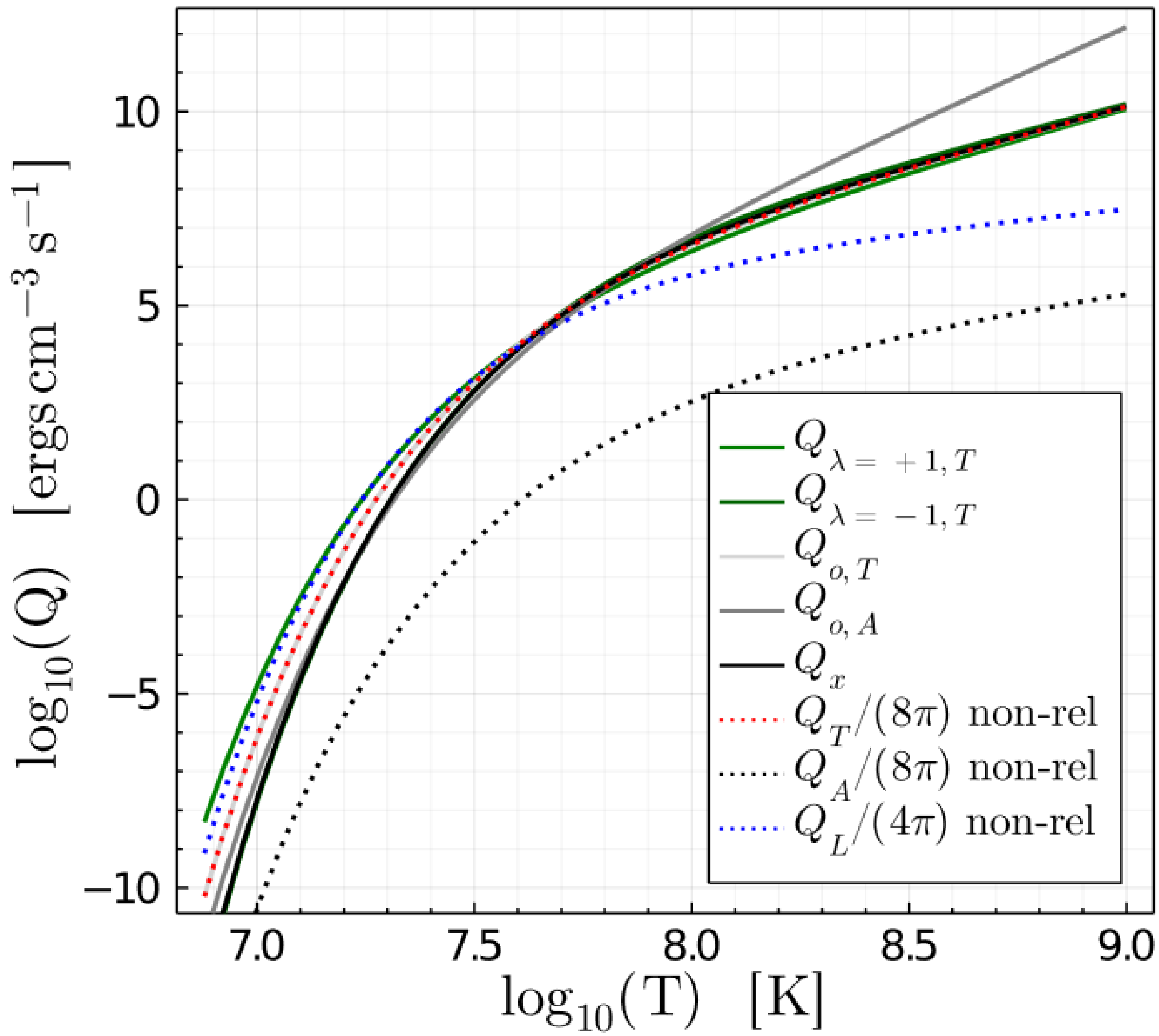
- Magnetic fields force electrons on Landau levels, modify refractive index

$$E_\nu = \sqrt{p_\parallel^2 + m_e^2\left(1 + 2\nu \frac{B}{B_c}\right)}$$

- Other effects (Schwinger-like pair creation, modification of wave function…) are negligible or sub-dominant

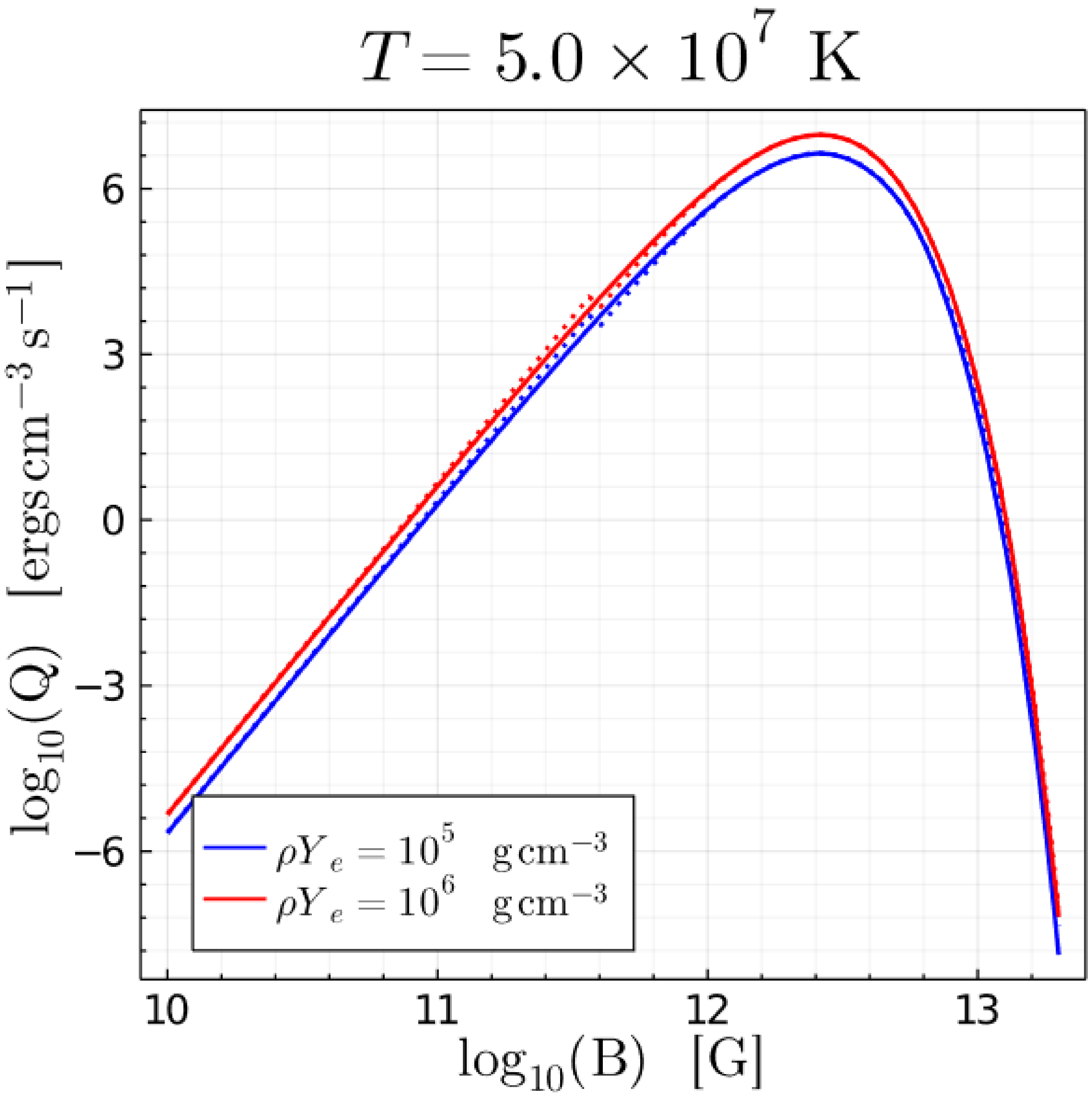
Plasma Processes

$\rho Y_e = 10^6\,\mathrm{g\,cm^{-3}}$



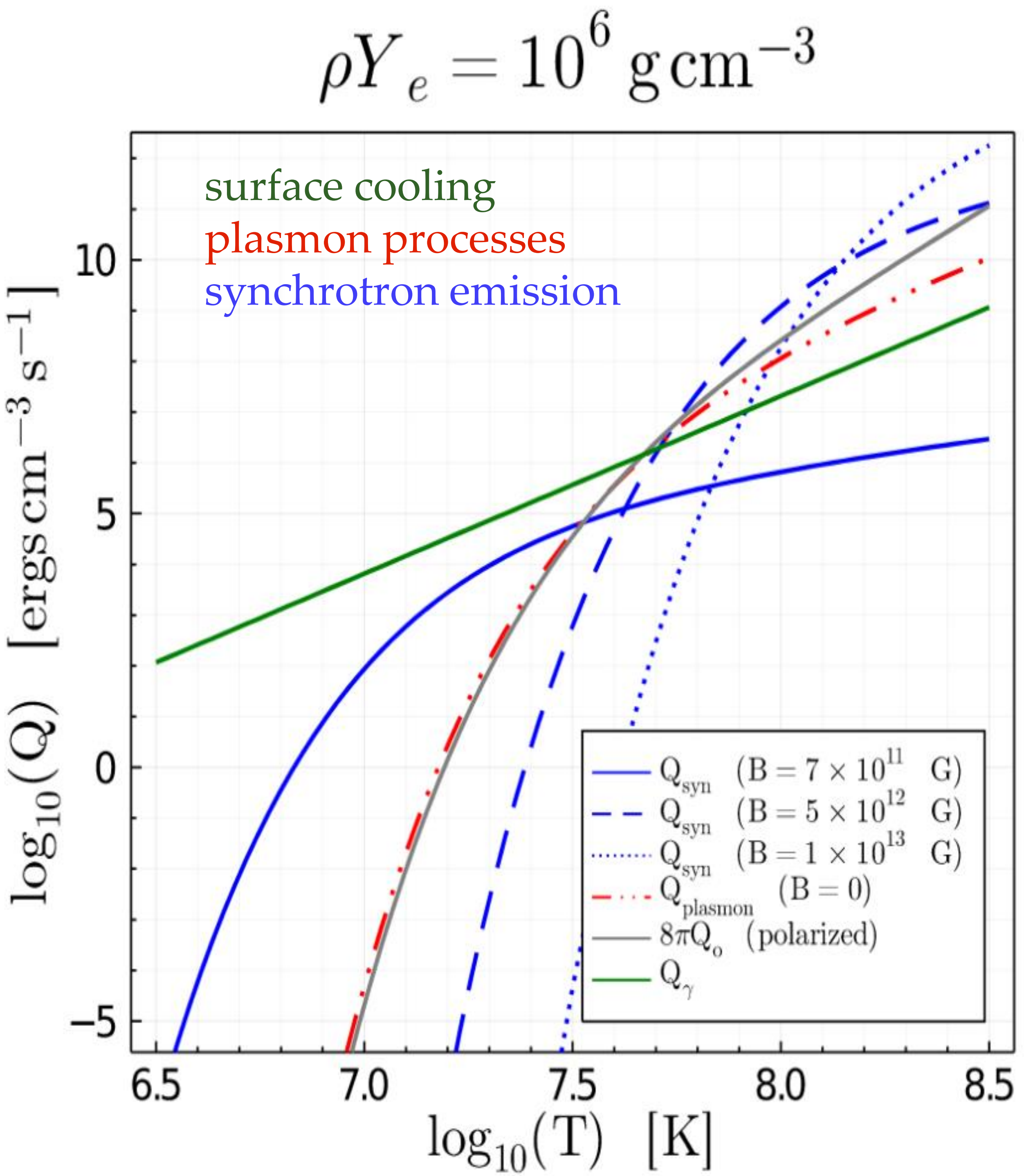
- For typical WD parameters, impact of B fields significant…
- …but only at temperature where other processes are more important

Synchrotron Radiation



- B-fields open up new cooling channel e → e**νν**
- In relevant regime the effect grows with B
- For very large B: suppression because next Landau level becomes inaccessible

Comparing Mechanisms



- Synchrotron emission can dominate for large temperatures
- Requires comparably large B fields
- Can potentially solve the anomaly for

$$B \sim 3 \times 10^{11}\ \mathrm{G}$$

- But how to generate these fields?
- Non-observation of stronger anomaly imposes upper bound

$$B < 6 \times 10^{11}\ \mathrm{G}$$