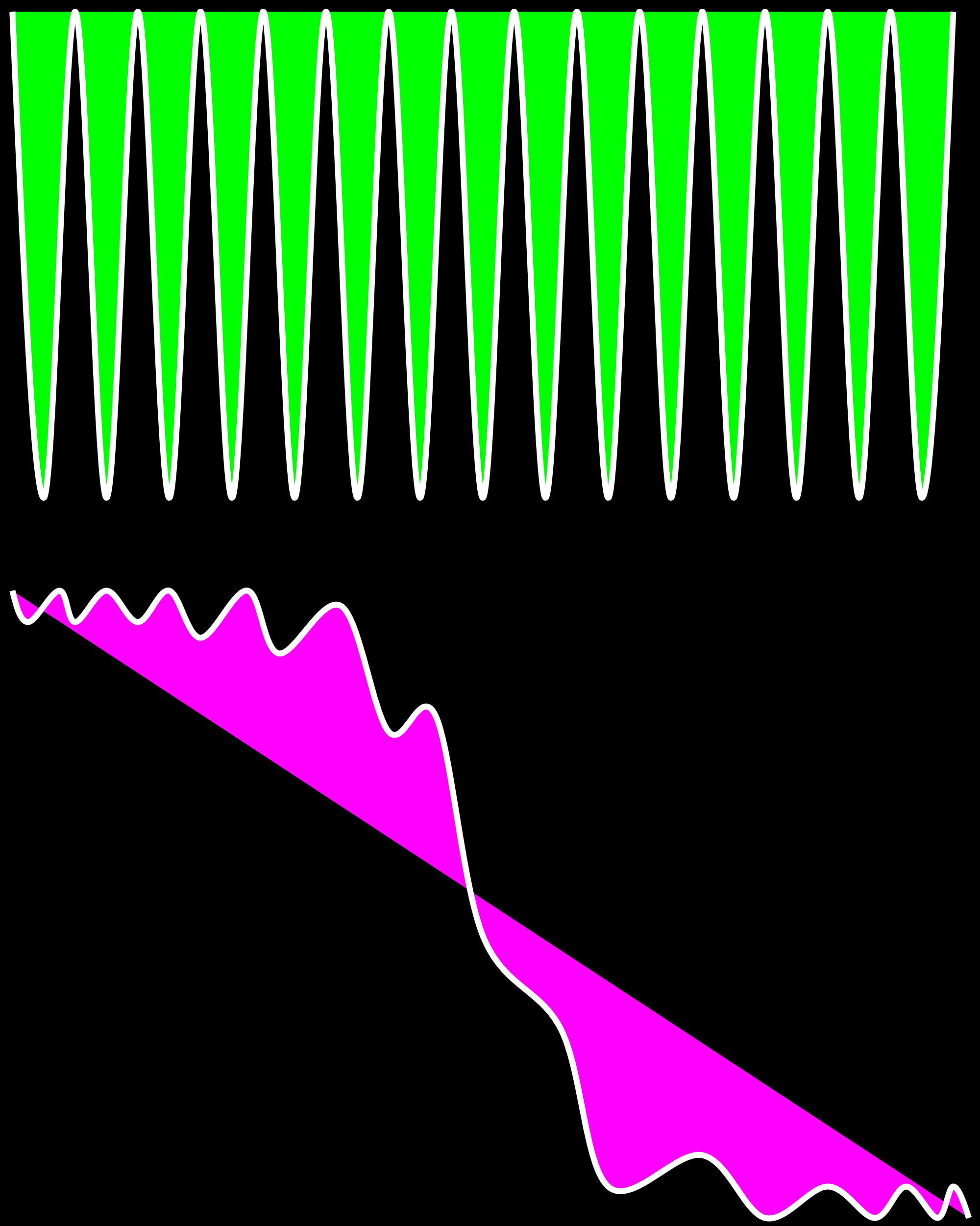
II. Matter effects



Content

1. Refraction and matter potential
2. Hamiltonian, eigenstates and mixing
3. Flavor conversion in different media
4. Collective oscillations

Evolution equation

If loss of coherence and other complications related to WP picture are irrelevant – `` point-like" picture. Evolution equation:

$$i \frac{d\Psi}{dt} = H\Psi$$

$$\Psi = \begin{pmatrix} \Psi_e \\ \Psi_\mu \\ \Psi_\tau \end{pmatrix}$$

For ultra relativistic ν

$$H = E \sim p + \frac{m^2}{2E}$$

omit p, substitute $m^2 \to M M^+$, where M is 3x3 mass matrix

$$H = \frac{M M^+}{2E}$$

$$M M^+ = U M_{diag}^2 U^+$$

$$M_{diag}^2 = \mathrm{diag}\,(m_1^2,\ m_2^2,\ m_3^2)$$

U - mixing matrix

Hamiltonian for 2ν

In the flavor basis $\nu_f = (\nu_e , \nu_\tau)^T$

$$H_{tot} = \frac{M M^+}{2E}$$

$$M M^+ = U M_{diag}^2 U^+$$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$M_{diag}^2 = \mathrm{diag}\,(m_1^2, \; m_2^2)$$

$$H_{tot} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Neutrino polarization vector in the flavor space

$\psi = \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix}$ ⇒ Polarization vector:

$$\mathbf{P} = \psi^+ \sigma/2\, \psi$$

Pauli matrices

$$\mathbf{P} = \begin{pmatrix} Re\, \nu_e^+ \nu_\tau \\ Im\, \nu_e^+ \nu_\tau \\ \nu_e^+ \nu_e - 1/2 \end{pmatrix}$$

elements of density matrix

Evolution equation:

$$i \frac{d\Psi}{dt} = H\Psi \quad \Rightarrow \quad i \frac{d\Psi}{dt} = -(\mathbf{B}\,\sigma/2)\,\Psi$$

where $\mathbf{B} = \frac{2\pi}{l_\nu}(\sin 2\theta, 0, \cos 2\theta)$

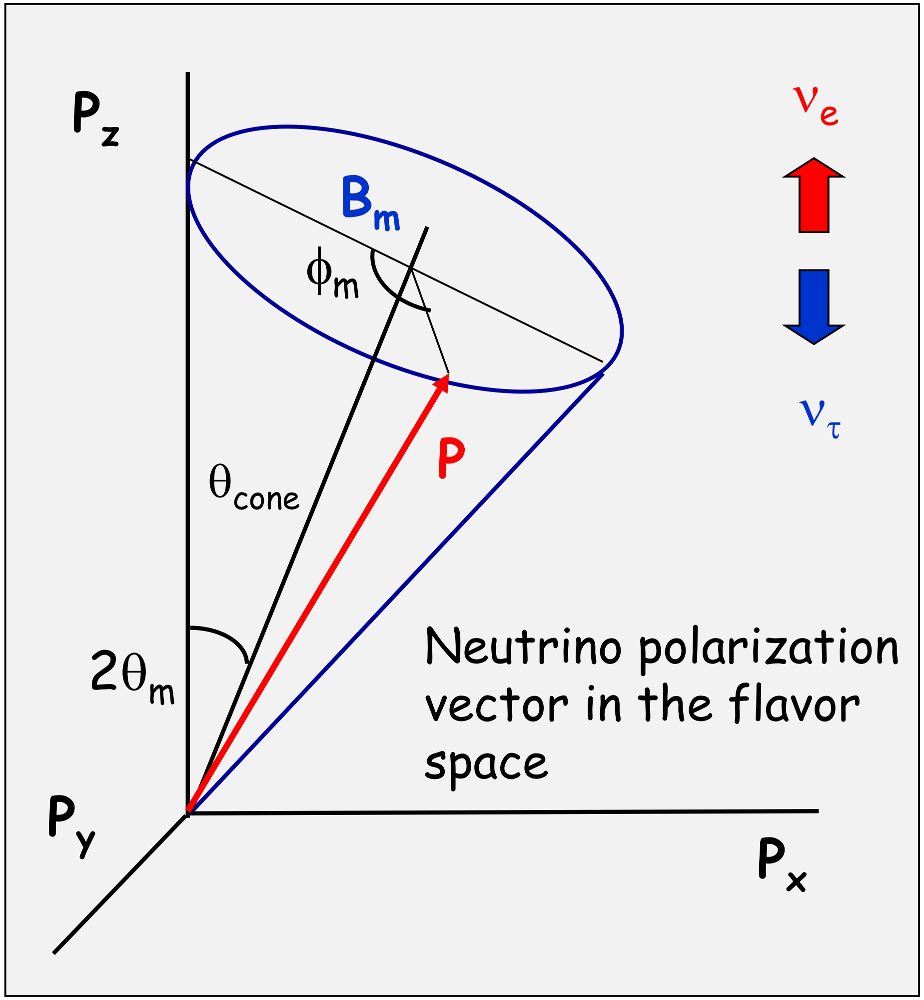
Differentiating **P** and using evolution equation for Ψ

$$\frac{d\mathbf{P}}{dt} = -[\mathbf{B} \times \mathbf{P}]$$

coincides with equation for the electron spin precession in the magnetic field

Graphic representation

of oscillations as the electron spin precession in the magnetic field



$|\mathbf{P}| = \frac{1}{2}$ polarization vector

$\mathbf{B} = \frac{2\pi}{l_\nu}(\sin 2\theta,\ 0,\ \cos 2\theta)$

$P_{ee} = \nu_e^+ \nu_e = P_Z + 1/2$

$\phi = 2\pi t / l_\nu$ oscillation phase

In general, degrees of freedom in matter:

θ_m (n, E) - mixing angle
ϕ_m (n, E) - phase
θ_{cone} (dn/dx) - cone angle

Subscript m means in medium for vacuum

1. Refraction
Matter potentials

Refraction

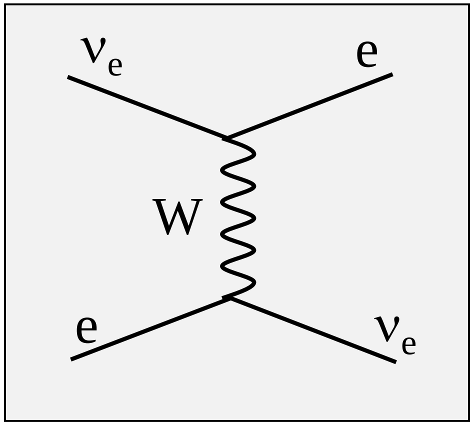
L. Wolfenstein, 1978

at low energies Re A >> Im A
inelastic interactions can be neglected

Elastic forward scattering → V_e, V_μ

potentials

for ν_e ν_μ



Refraction index:

$$n - 1 = V / p$$

difference of potentials

$$V = V_e - V_\mu = \sqrt{2}\, G_F\, n_e$$

for E = 10 MeV

$$n - 1 = \begin{cases} \sim 10^{-20} & \text{inside the Earth} \\ < 10^{-18} & \text{inside the Sun} \end{cases}$$

$V \sim 10^{-13}$ eV inside the Earth

Refraction length:

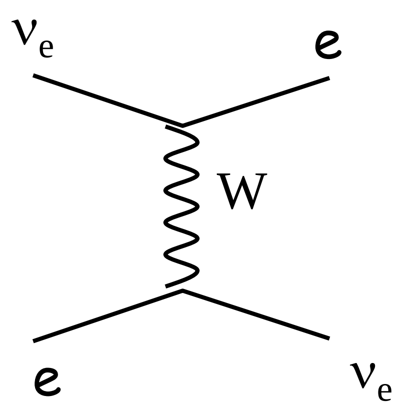
$$l_0 = \frac{2\pi}{V}$$

Matter potential computations

$$H_{int}(\nu) = \langle \psi | H_{int} | \psi \rangle = V \; \nu^{+} \nu$$

ψ - wave function of medium

for CC interactions with electrons: $|\psi\rangle = |e\rangle$



after Fierz transformation

$$H_{int} = \frac{G_F}{\sqrt{2}} \, \overline{\nu} \gamma^{\mu} (1 - \gamma_5) \nu \, \overline{e} \gamma_{\mu} (1 - \gamma_5) e$$

specific matrix elements:

$\langle e | \gamma_0 (1 - \gamma_5) | e \rangle = n_e$ electron number density → $V = \sqrt{2}\, G_F n_e$

For unpolarized medium at rest

$\langle e | \boldsymbol{\gamma} | e \rangle = n_e \mathbf{v}$ - velocity of electrons

$\langle e | \boldsymbol{\gamma} \gamma_5 | e \rangle = n_e \boldsymbol{\lambda}_e$ - averaged polarization vector of electrons

Matter potential

Essentially classical consideration: electrons generate classical field interacting with neutrinos

Potential can be derived by summation (integration) of Yukawa potentials with radius $1/m_W$ produced by individual electrons

$$V(\mathbf{r}) = 2\frac{g^2}{8}\int_0^\infty d\mathbf{x}\, n_e(\mathbf{x})\, \frac{1}{4\pi\, |\mathbf{r} - \mathbf{x}|}\, e^{-m_W |\mathbf{r} - \mathbf{x}|}$$

For uniform density distribution after integration over angular variables

$$V = 2\int_0^\infty dr\, r^2\, n_e \frac{g^2}{8r}\, e^{-m_W r}$$

This requires uniform distribution of e in the radius of several $1/m_W$

The integration gives $V_e = 2\sqrt{G_F n_e}$

From micro to macro picture

From interactions with individual scatteres to effective potential (mean field approximation)

E.Kh. Akhmedov 2010.07847 [hep-ph]

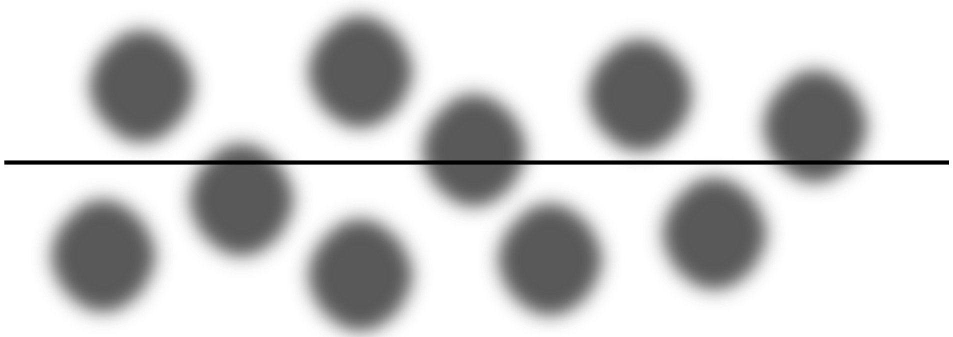
Point-like scatterers, a coarse graining – coordinate space averaging over macroscopic volumes with large number of particles

A. Y.S. , Xun-jie Xu

Summation of potentials produced by individual scatteres.

e.g., G Fantini, A.G. Rosso, F. Vissani 1802.05781

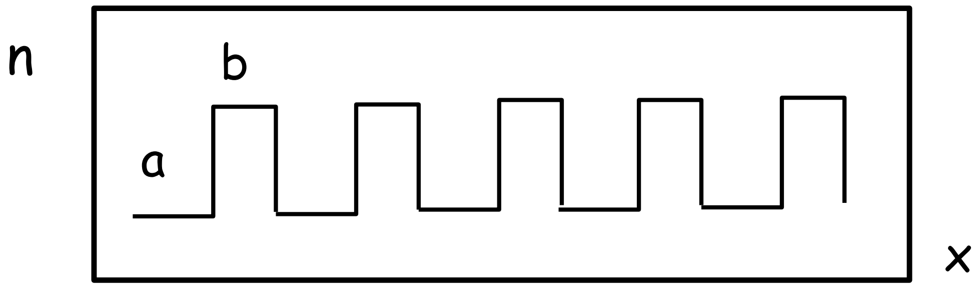
For short range interactions r_{WI}, localization of scatterers should be taken into account $X_e >> r_{WI}$, e.g. localization of e in atom



since $\lambda_\nu \sim 1/p_\nu << X_e$

→ make sense to consider propagation of neutrino inside atom?

Modeling with castle wall profile



V_a V_b Half – phases: ϕ_a ϕ_b

L_a L_b Mixing angles: θ_a θ_b

Oscillation probability

E. Kh. Akhmedov

$$P = [1 - I^2/(1 - R^2)] \sin^2 (n\,\zeta)$$

ζ = arcos R   n – number of periods

$$I = I(\phi_a, \phi_b, \theta_a, \theta_b), \quad R = R(\phi_a, \phi_b, \theta_a, \theta_b)$$

For ϕ_a ϕ_b << 1 the probability can be reduced to

$$P = \sin^2 2\theta_m(\bar{V}) \sin^2 \tfrac{1}{2}\phi(\bar{V})$$

$$\bar{V} = \frac{V_a L_a + V_b L_b}{L_a + L_b}$$

- averaged potential

2. Hamiltonian in matter Eigenstates and mixing

instantaneous

Hamiltonian in matter

Add to energy in vacuum the potential which describes interaction of neutrino with matter

$$H(n_e, E) = H_0 + V$$

$$V = \mathrm{diag}(V_e, 0, 0)$$

$$V_e = \sqrt{2}\, G_F n_e$$

in the flavor basis

This can be obtained in a systematic way starting from Lagrangian for neutrinos, electrons and W-bosons.

... find EOM, take non-relativistic electrons, use zero component of current which give density of electrons

Eigenstates in matter

Eigenstates of the Hamiltonian in matter:

$$H\,(n_e, E\,) = H_0 + V\,(n_e)$$

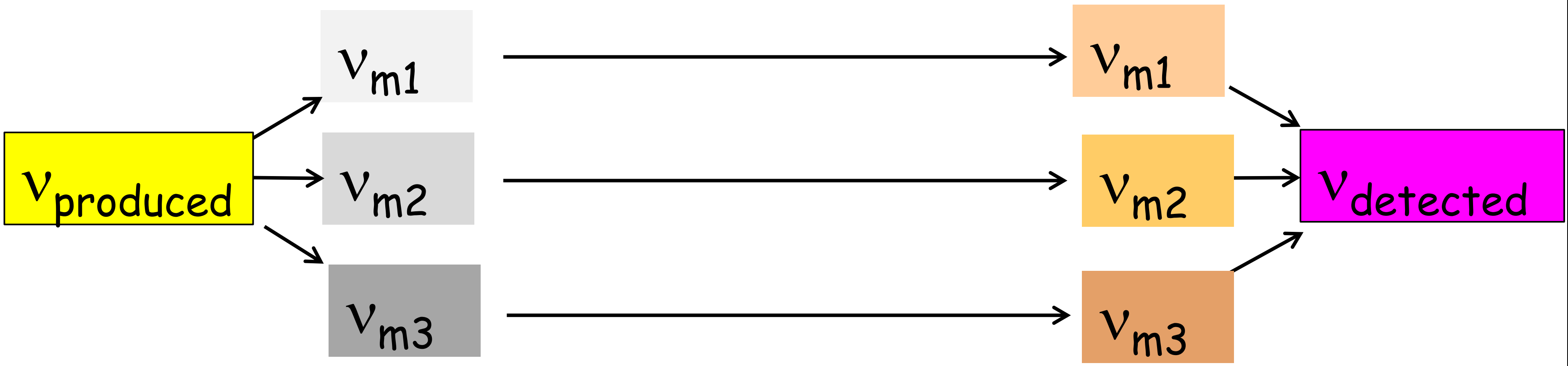
Play the same role as mass states in vacuum

$$\nu_k \rightarrow \nu_{mk} \qquad\qquad k = 1, 2, 3$$

In matter with constant density ν_{mk} propagate independently.

Formally, they diagonalize Hamiltonian H → in the basis of ν_{mk} the Hamiltonian is diagonal. Consequently, equation of motions for them split.

Evolution in terms of eigenstates in matter



Decompose initial state into eigenstates of propagation

By definition they propagate independently

Transitions between them are negligible or small

Still properties (flavors) of the eigenstates may change

Projection onto detected state

if density varies

Flavor mixing in matter

Mixing in matter is determined with respect to eigenstates in matter

Mixing matrix in matter connects the flavor states with eigenstates in matter:

$$\nu_f = U^m \nu_m$$

in the same way as in vacuum the PMNS matrix connects the flavor states with mass states:

$$U_{PMNS} \rightarrow U^m(n_e, E)$$

U^m diagonalizes the Hamiltonian in matter

$$U^{m+} H U^m = H^{diag}$$

$$H^{diag} = diag(H_{1m}, H_{2m}, H_{3m})$$

eigenvalues of the Hamiltonian

Inverting relation:

$$\nu_m = U^{m+}(n_e, E)\, \nu_f$$

which means that flavor composition of eigenstates depends on n_e and E

Hamiltonian, eigenstates and eigenvalues

$$H(n_e, E) = H_0 + V$$

$$V = \mathrm{diag}(V_e,\ 0)$$

In the flavor basis, $(\nu_e, \nu_\mu)^T$

$$H_{tot} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta + \xi & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \qquad \begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix}$$

$$\xi = \frac{4 V_e E}{\Delta m^2} \qquad V_e = \sqrt{2}\, G_F n_e$$

Mixing in matter

Diagonalization of the Hamiltonian:

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}$$

Mixing is maximal $\sin^2 2\theta_m = 1$, if

$$V = \frac{\Delta m^2}{2E} \cos 2\theta$$

Resonance condition

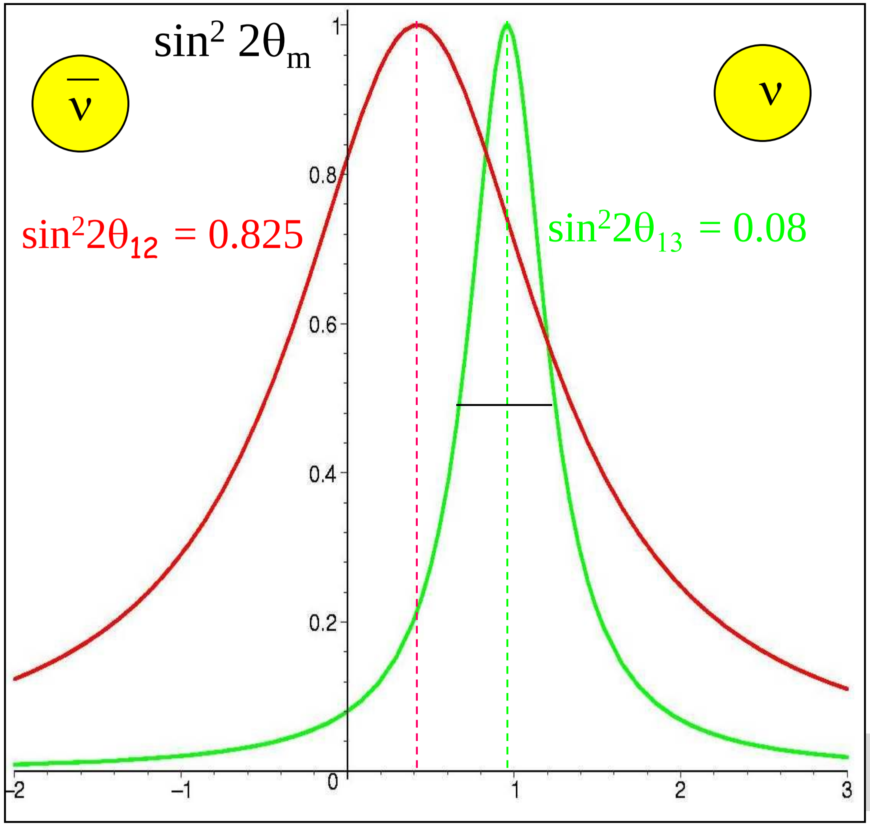
$$H_e = H_\mu$$

Difference of the eigenvalues

$$H_{2m} - H_{1m} = \frac{\Delta m^2}{2E} \sqrt{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}$$

Resonance

Dependence of mixing on density, energy has a resonance character



In resonance: $\sin^2 2\theta_m = 1$

→ flavor mixing is maximal

$l_\nu = l_0 \cos 2\theta$

Vacuum oscillation length ≈ Refraction length

Resonance width: $\Delta n_R = 2 n_R \tan 2\theta$

density $l_\nu / l_0 \sim n E$

$\theta_m \rightarrow 0$ | $\theta_m = \theta$ | $\theta_m = \pi/4$ | $\theta_m \rightarrow \pi/2$

Mixing is suppressed at high densities

$$|V| \gg \frac{\Delta m^2}{2E}$$

Flavor states coincide with eigenstates and vice versa

Level crossing

V. Rubakov, private comm.
N. Cabibbo, Savonlinna 1985
H. Bethe, PRL 57 (1986) 1271

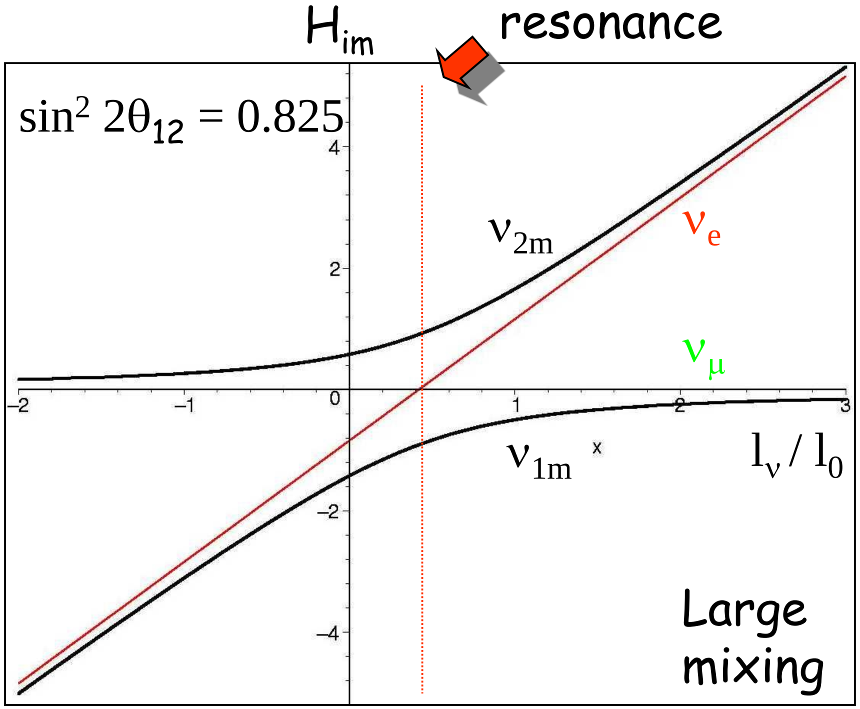
Dependence of the eigenvalues on the matter potential (density):

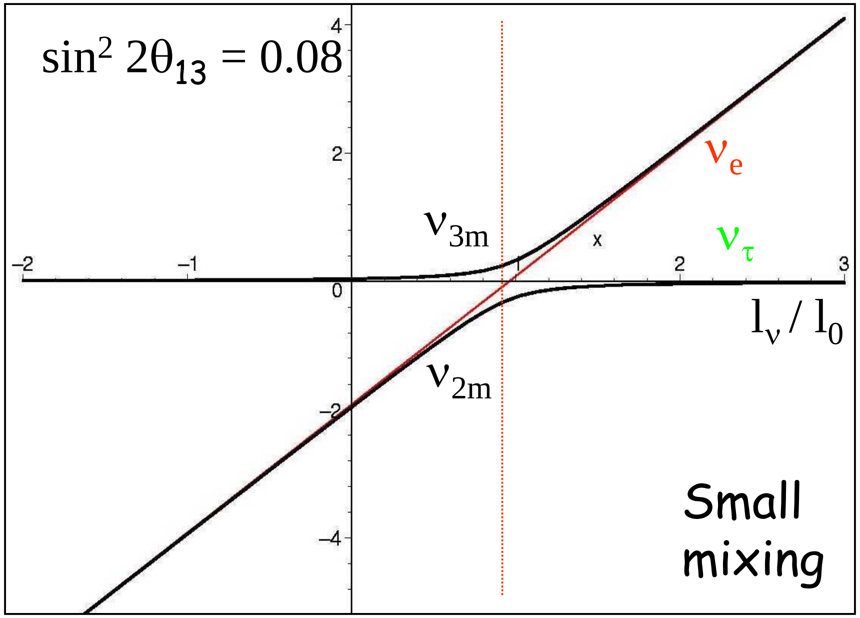
$$\frac{l_\nu}{l_0} = \frac{2E\,V}{\Delta m^2}$$

Crossing point - resonance

$$\frac{l_\nu}{l_0} = \cos 2\theta$$

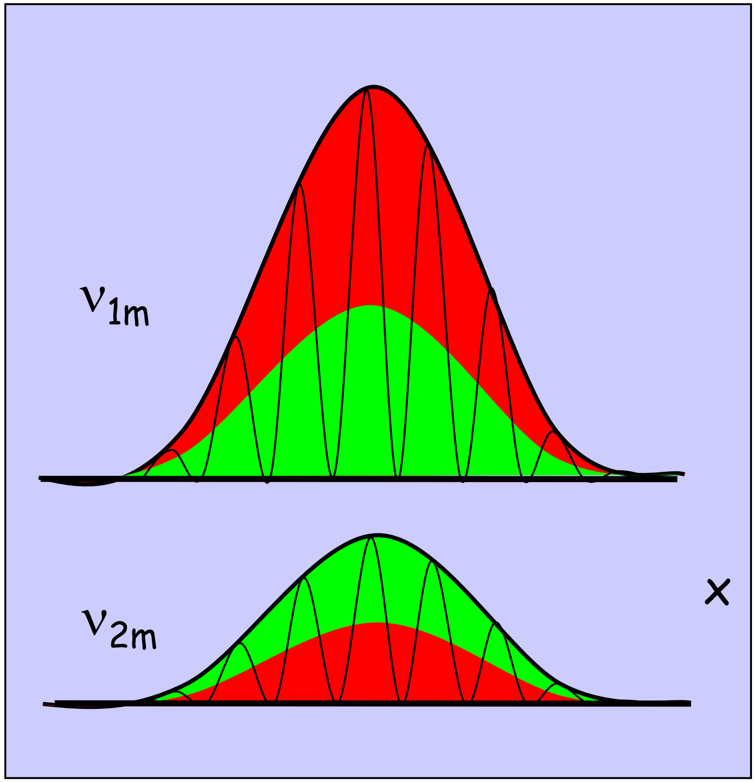
In the crossing point
- the level split is minimal
- the oscillation length is maximal





3. Flavor convertion in different media

Oscillations in matter



Constant density medium:
the same dynamics as in vacuum

$\nu_k \rightarrow \nu_{mk}$

Mixing changed

$\theta \rightarrow \theta_m(n)$

Difference of eigenvalues changed

$\frac{\Delta m^2}{2E} \rightarrow \Delta H$

→ difference of phase velocities changed

$$V = \cos 2\theta \frac{\Delta m^2}{2E}$$

Oscillations in matter

Probability in constant density

$$P(\nu_e \rightarrow \nu_a) = \sin^2 2\theta_m \sin^2\left(\frac{\pi L}{l_m}\right)$$

depth of oscillations — oscillatory factor — half-phase ϕ

$\theta_m(E, n)$ - mixing angle in matter

$l_m(E, n)$ – oscillation length in matter

$$l_\nu \rightarrow l_m$$

$$l_m = 2\pi/(H_{2m} - H_{1m})$$

Maximal effect: $\sin^2 2\theta_m = 1$

$\phi = \pi/2 + \pi k$ ➡ MSW resonance condition

Resonance enhancement

For neutrinos propagating in the mantle of the Earth

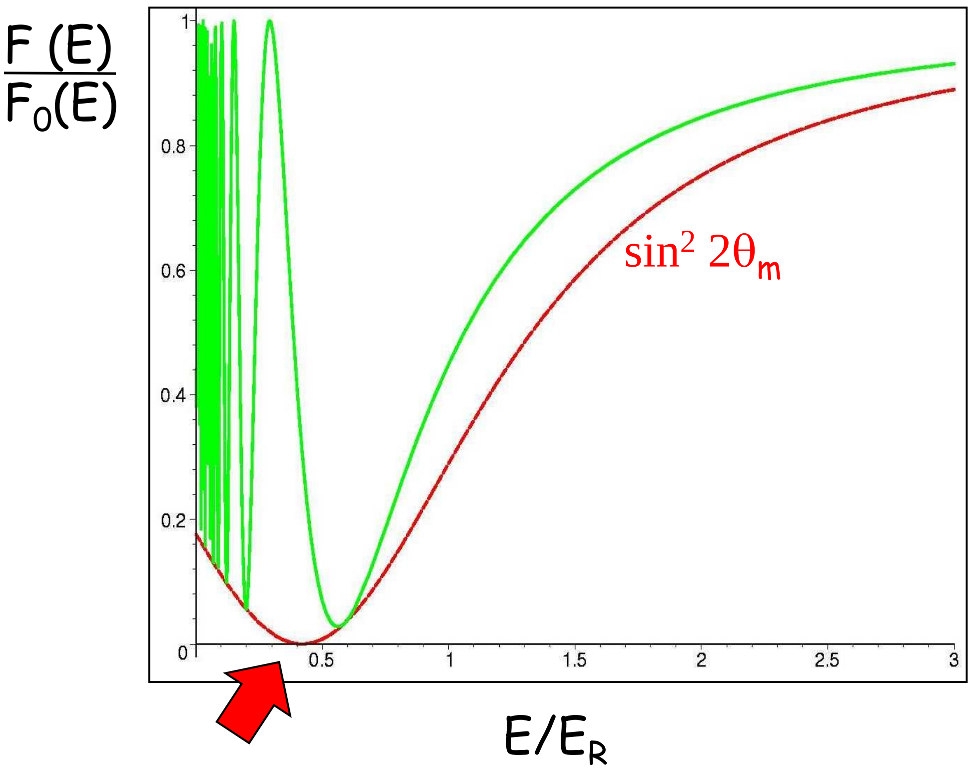
Survival probability

Large mixing $\sin^2 2\theta = 0.824$

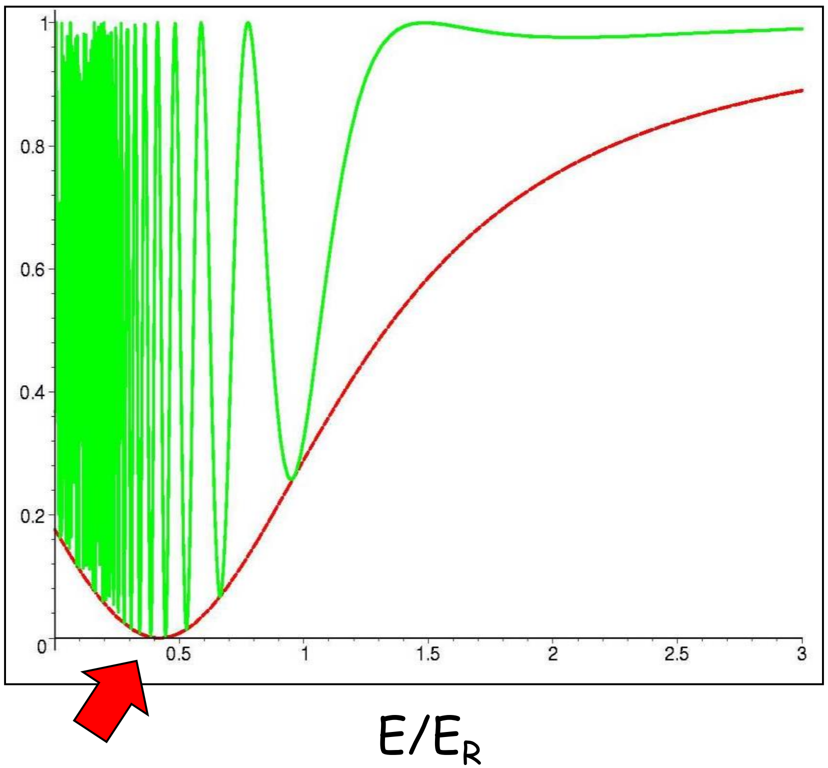
Layer of length L

$k = \pi L / l_0$

thin layer $k = 1$



thick layer $k = 10$



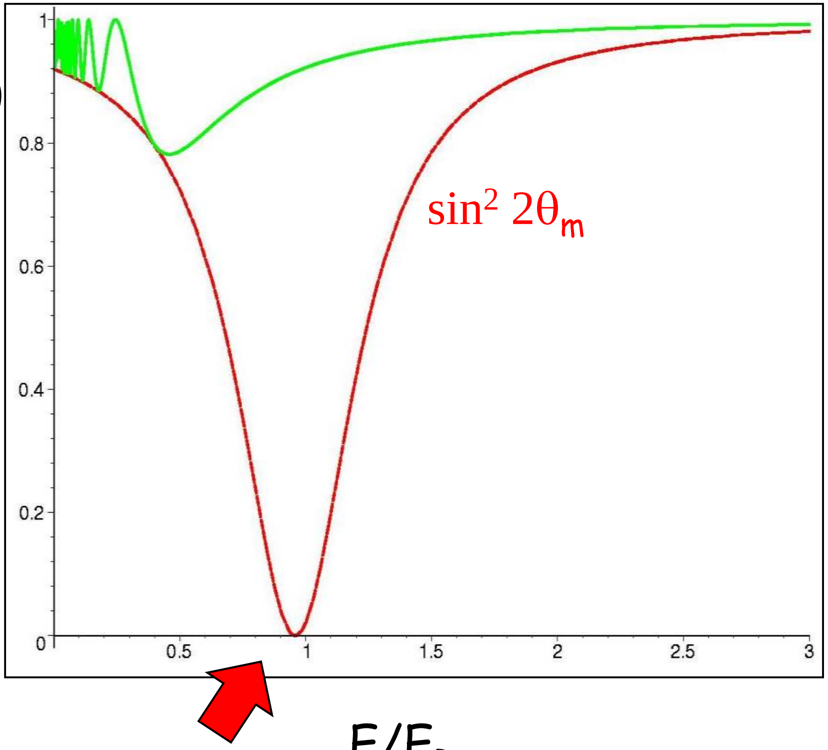
Oscillation in the Earth, ORCA

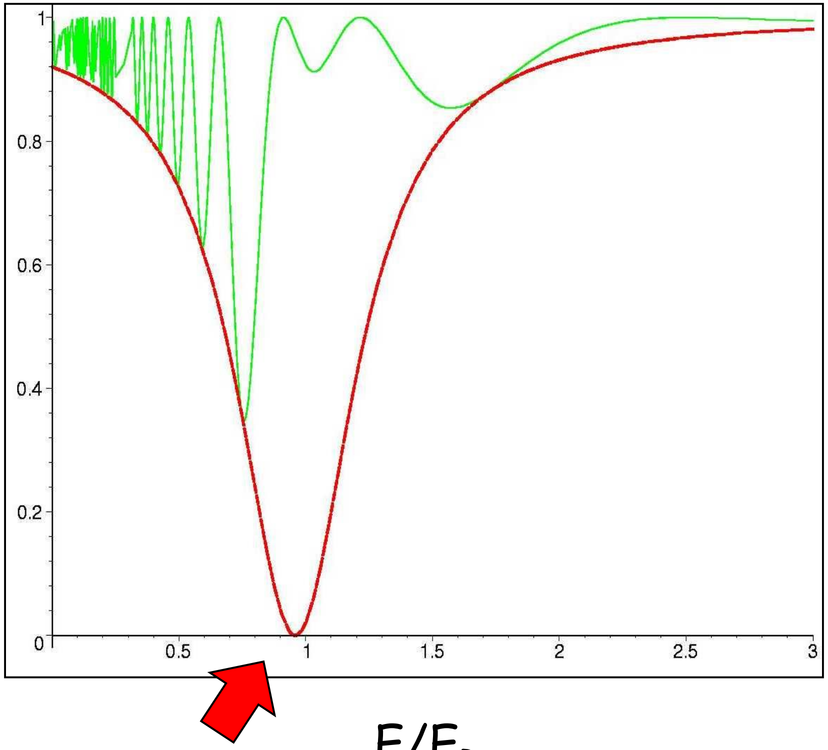
Small mixing $\sin^2 2\theta = 0.08$

thin layer $k = 1$

thick layer $k = 10$

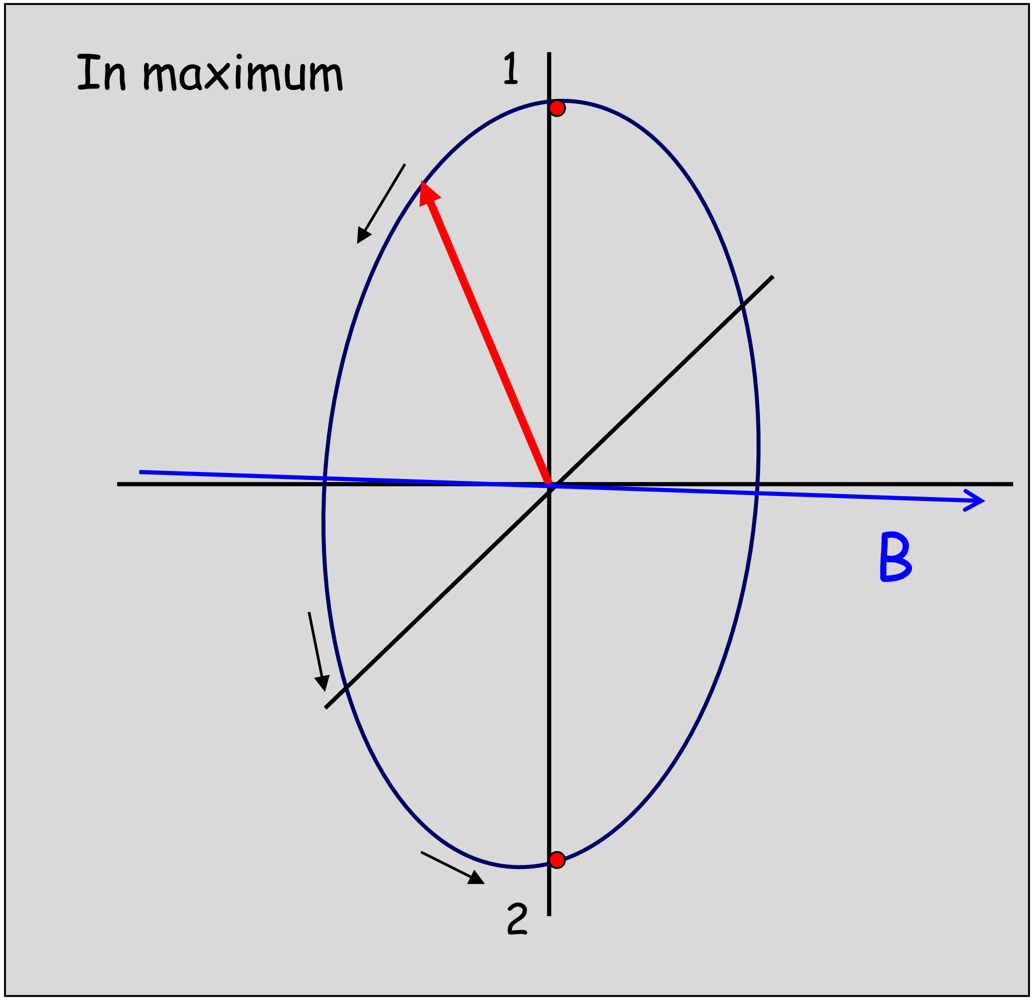
$\frac{F(E)}{F_0(E)}$

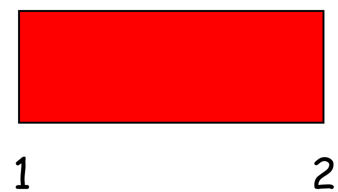




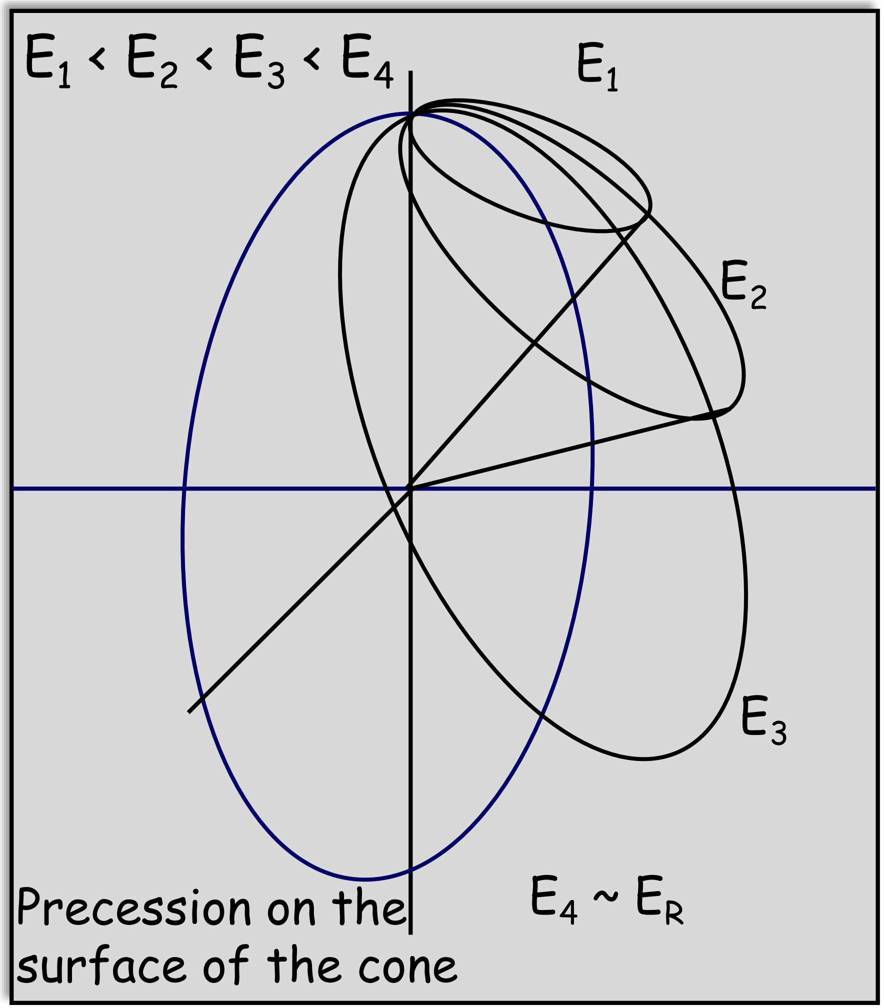
E/E_R E/E_R

Resonance enhancement





Graphic representation



Resonance enhancement

Adiabatic conversion

The MSW effects

- the flavor transformations driven by dependence of the mixing in matter on density or/and energy of neutrino.

$$\nu_f = U^m(E, n)\, \nu_H$$

Flavor states

Mixing matrix in matter

Eigenstates of Hamiltonian in mater - Eigenstates of propagation

Mixing angles $\theta^m(E, n)$ become dynamical variables in contrast, to mixing in vacuum

Dependence of $\sin^2 2\theta^m$ on E and n has resonance character

Evolution equation for eigenstates. Adiabaticity

Varying density

Inserting in evolution equation for the flavor states $\nu_f = U^m \nu_m$

$$i \frac{d\nu_m}{dt} = \left(H^{diag} + i\, U^{m+} \frac{dU^m}{dt} \right) \nu_m$$

$$H^{diag} = diag(H_{1m}, H_{2m}, H_{3m})$$

off - diagonal

If density changes slowly enough, so that

$$\left(U^{m+} \frac{dU^m}{dt} \right)_{ij} << H_{im} - H_{jm}$$

adiabaticity condition
Slow change of mixing

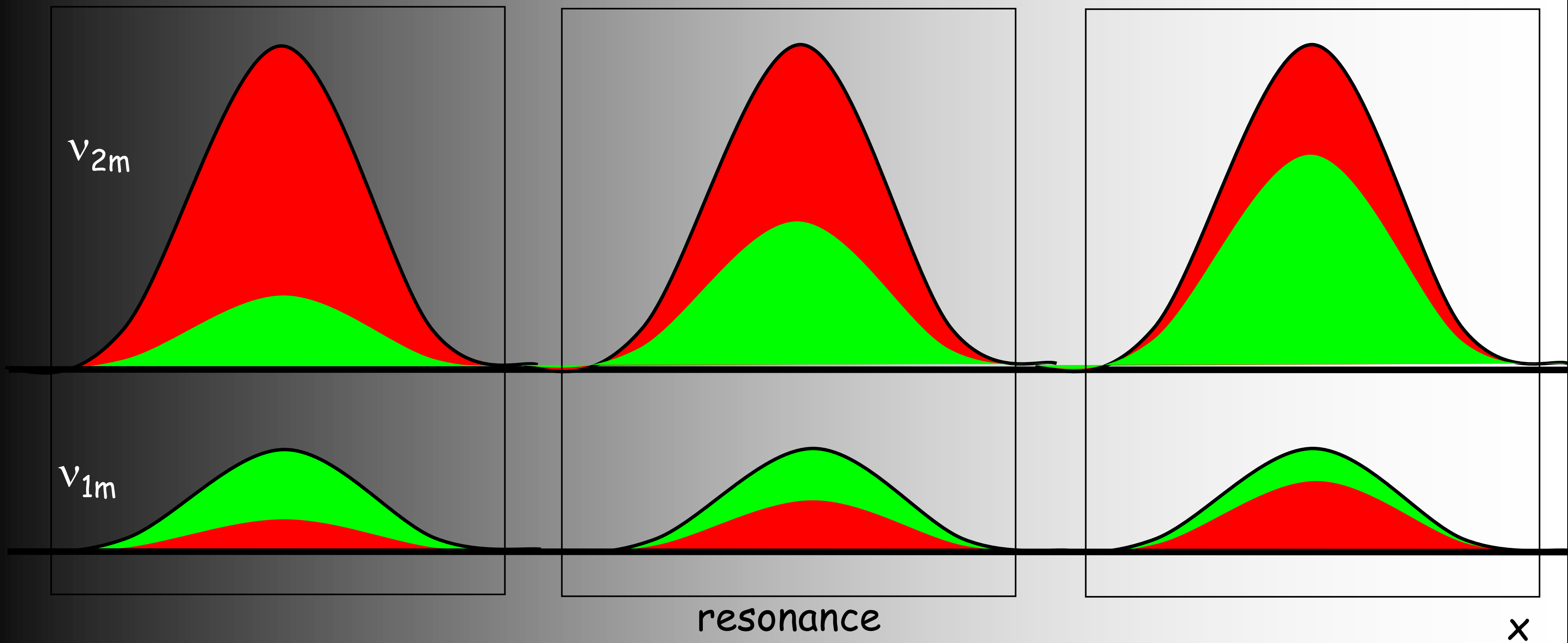
equation for the eigenstates splits:

$$i \frac{d\nu_m}{dt} = H^{diag} \nu_m$$

The eigenstates evolve independently, transitions $\nu_{im} \leftrightarrow \nu_{jm}$ are absent as in constant density.

In contrast to constant density case, the flavors of ν_{im} change according to density change

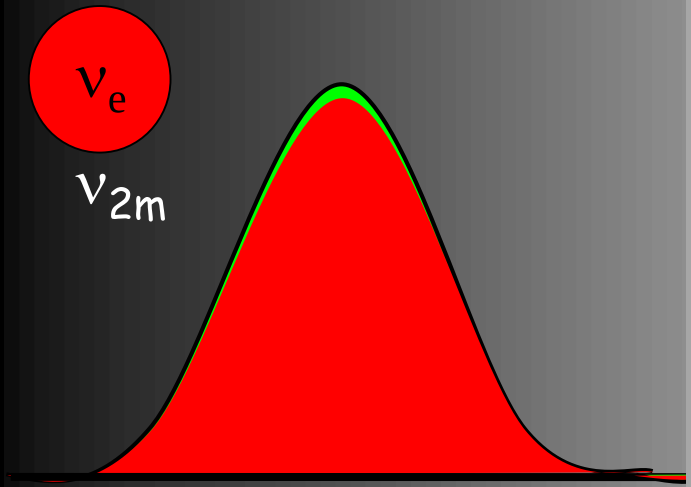
Adiabatic conversion



if density changes slowly

- the amplitudes of the wave packets do not change
- flavors of the eigenstates being determined by mixing angle follow the density change

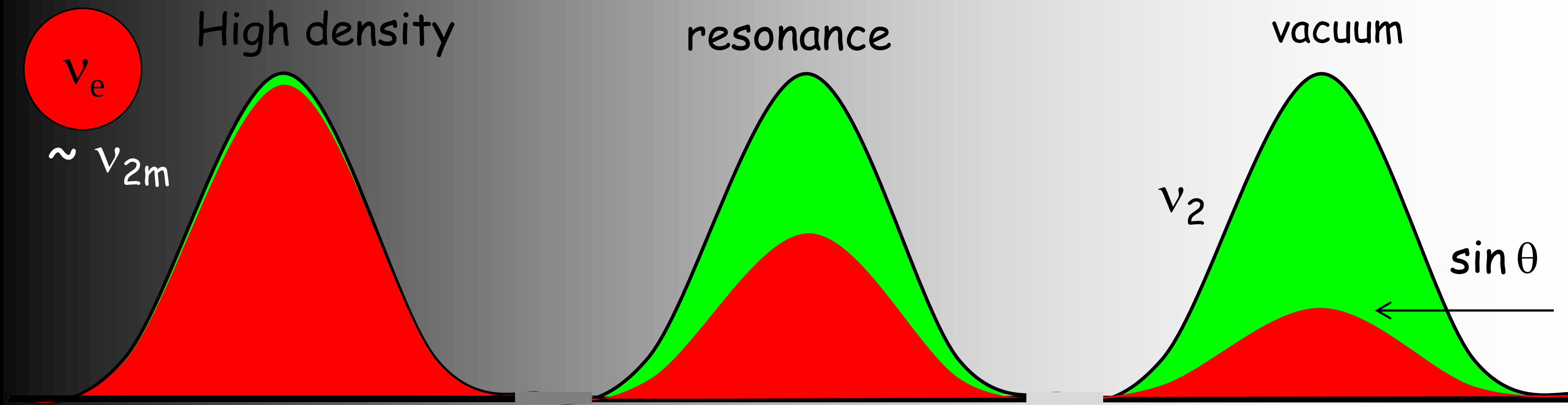
Non-oscillatory transition



Single eigenstate:
→ no interference
→ no oscillations
→ phase is irrelevant

This happens when mixing is very small in matter with very high density

Non-oscillatory transition



$\langle \nu_e | \nu_2 \rangle = \sin\theta$ x

mixing is very small

Single eigenstate:
→ no interference
→ no oscillations
→ phase is irrelevant

Survival probability $P_{ee} = \sin^2\theta$

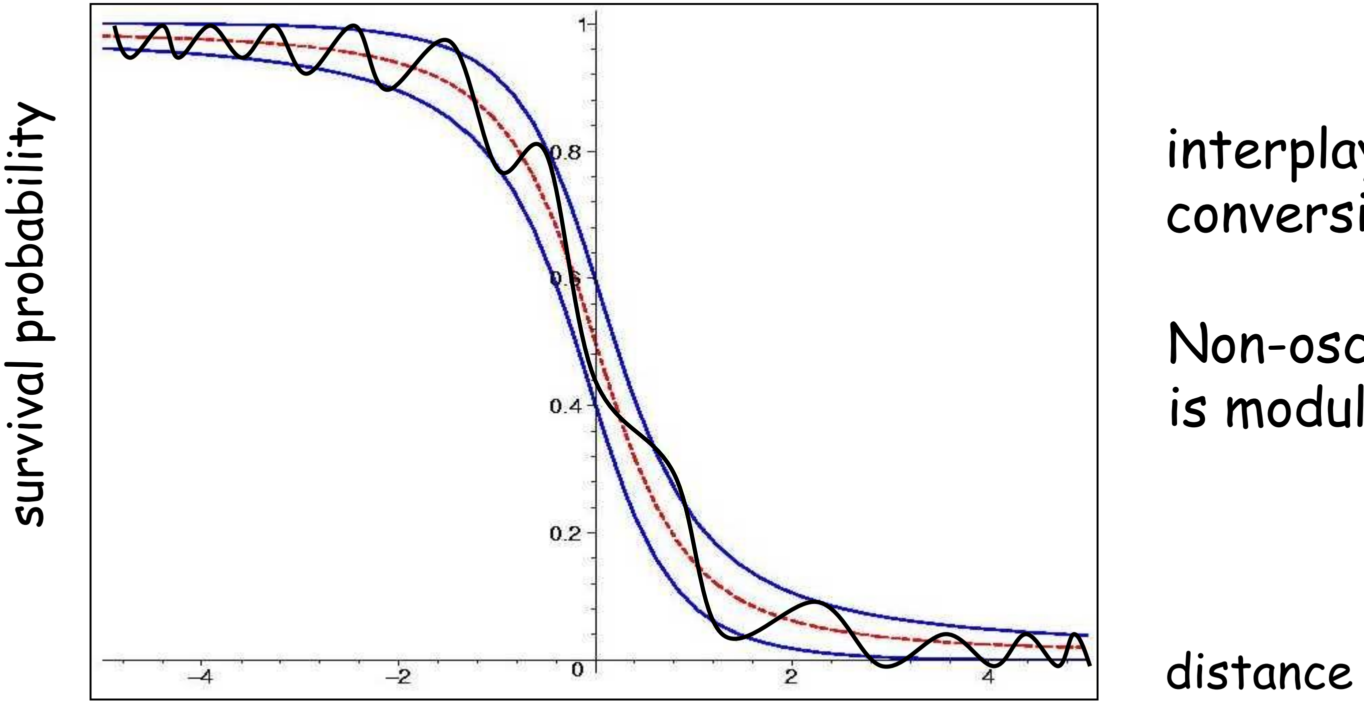
if density changes slowly (adiabatically)
→ no other eigenstate appear

$$\nu_{2m} \rightarrow \nu_2$$

Mixing and therefore the flavor content changes according to density change

Spatial picture

Adiabatic conversion



interplay of adiabatic conversion and oscillations

Non-oscillatory transition is modulated by oscillations

Adiabatic conversion probability

Sun, Supernova

Initial state: $\nu(0) = \nu_e = \cos\theta_m^0 \, \nu_{1m}(0) + \sin\theta_m^0 \, \nu_{2m}(0)$

Mixing angle in matter in production point

Adiabatic evolution to the surface of the Sun (zero density):

$\nu_{1m}(0) \rightarrow \nu_1$

$\nu_{2m}(0) \rightarrow \nu_2$

Final state: $\nu(f) = \cos\theta_m^0 \, \nu_1 + \sin\theta_m^0 \, \nu_2 \, e^{i\phi}$

Probability to find ν_e averaged over oscillations

$$P_{ee} = |\langle\nu_e|\nu(f)\rangle|^2 = (\cos\theta \cos\theta_m^0)^2 + (\sin\theta \sin\theta_m^0)^2$$

$$= 0.5[\, 1 + \cos 2\theta_m^0 \cos 2\theta \,]$$

or $P_{ee} = \sin^2\theta + \cos 2\theta \cos^2\theta_m^0$

Adiabaticity violation

If density $n_e(t)$ changes fast $\left|\frac{d\theta_m}{dt}\right| \sim |H_{2m} - H_{1m}|$

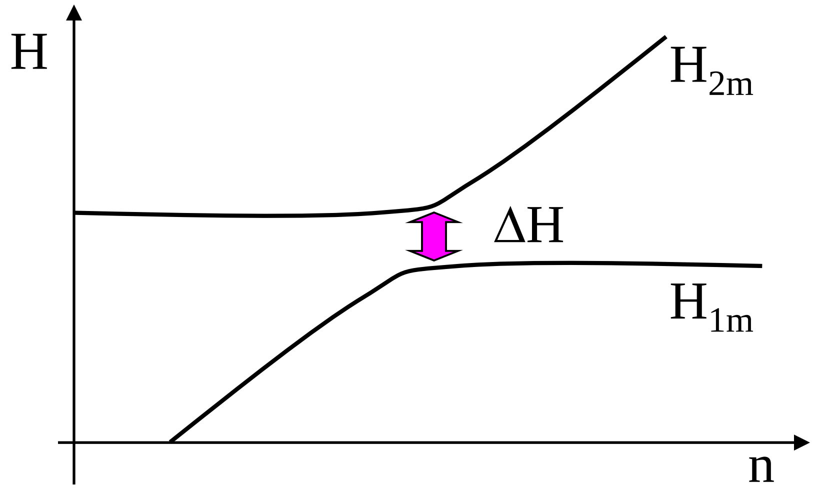
the off-diagonal terms in the Hamiltonian can not be neglected

transitions $\nu_{1m} \Leftrightarrow \nu_{2m}$

Admixtures of ν_{1m} ν_{2m} in a given propagating neutrino state change

``Jump probability''= penetration under barrier:

$$P_{12} = e^{-\frac{\Delta H}{E_n}}$$



$E_n = d\theta_m/dt \sim 1/h_n$

is the energy associated to change of density

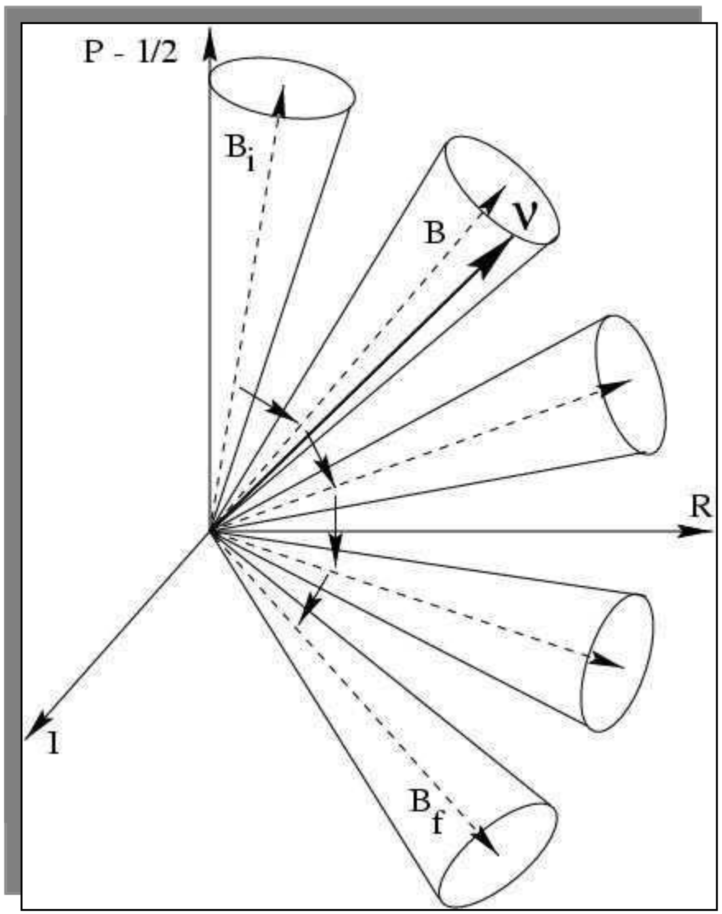
$$P_{12} = e^{-2\pi/\gamma_R}$$

Landau-Zenner

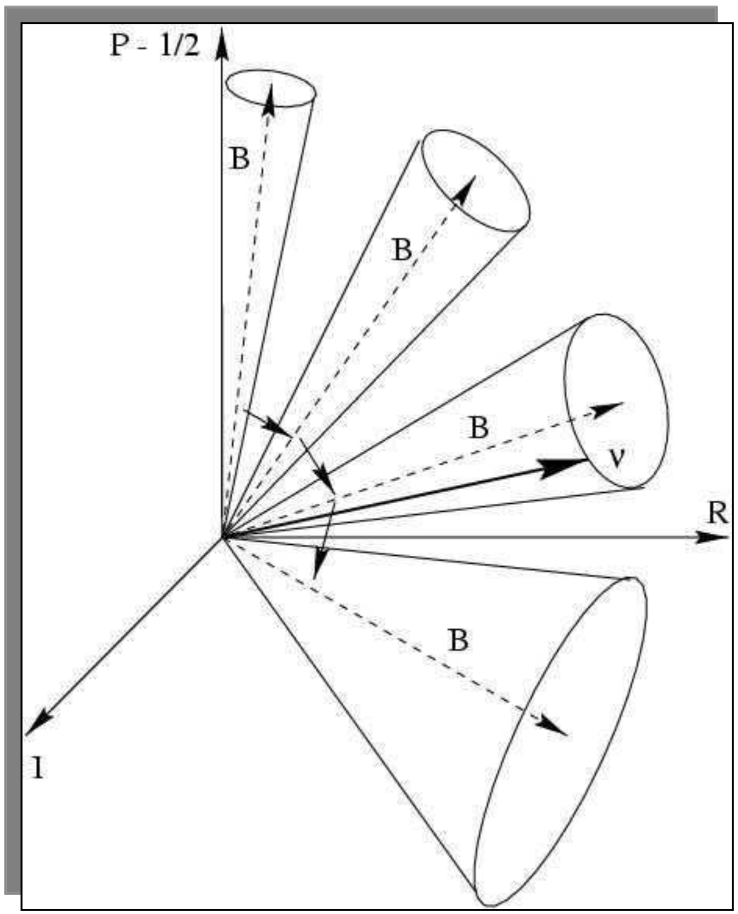
γ_R - adiabaticity parameter

Adiabatic conversion

Pure adiabatic conversion



Partially adiabatic conversion



Parametric effects

Propagation on the Earth
Wavy field
In central parts of SN
in neutrino field –collective oscilations

Parametric enhancement

V. Ermilova ,V. Tsarev, V. Chechin, Krat. Soob. Fiz. # 5, 26, (1986)

Strong transition if there is a harmonic modulation of density profile

$$n(x) = \langle n \rangle + n_1 \cos \omega_d x$$

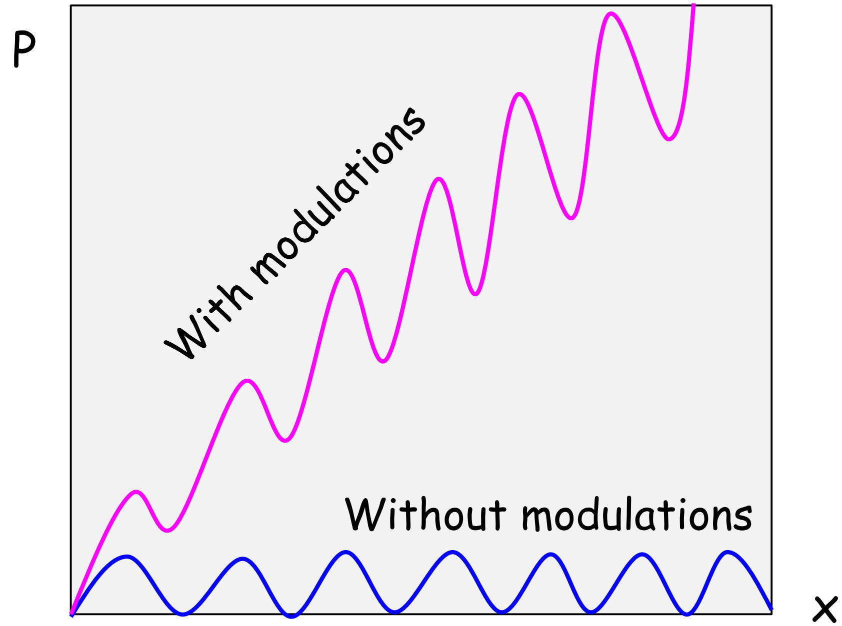
Parametric resonance condition:

$$k\,\omega_d = \Delta_m(\langle n \rangle)\ ,\ \ k = 1, 2\ \ldots$$

$$\Delta_m(\langle n \rangle) = \frac{\Delta m^2}{2E}\ \left[(\cos 2\theta - 2VE/\Delta m^2)^2 + \sin^2 2\theta\ \right]^{\frac{1}{2}}$$

is frequency of oscillations for the average density

Realized in astrophysical objects?

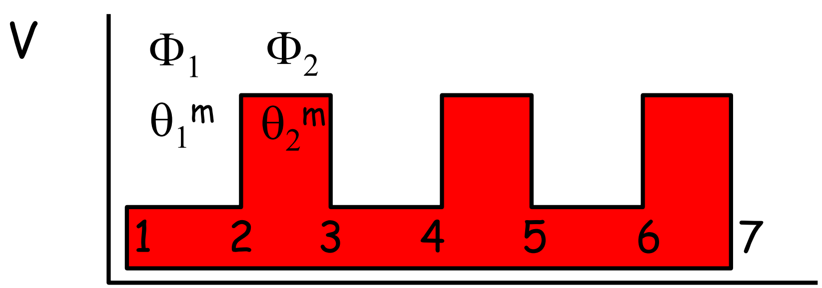


Parametric enhancement of oscillations

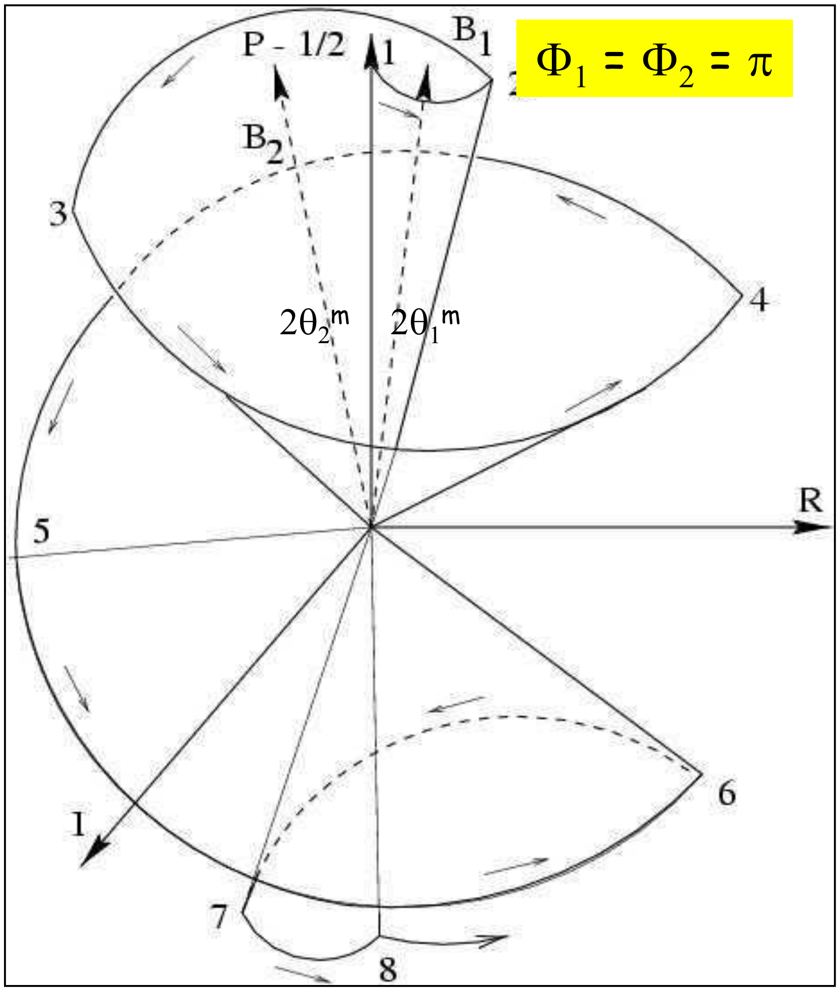
Parametric oscillations

E. Kh. Akhmedov , 1988

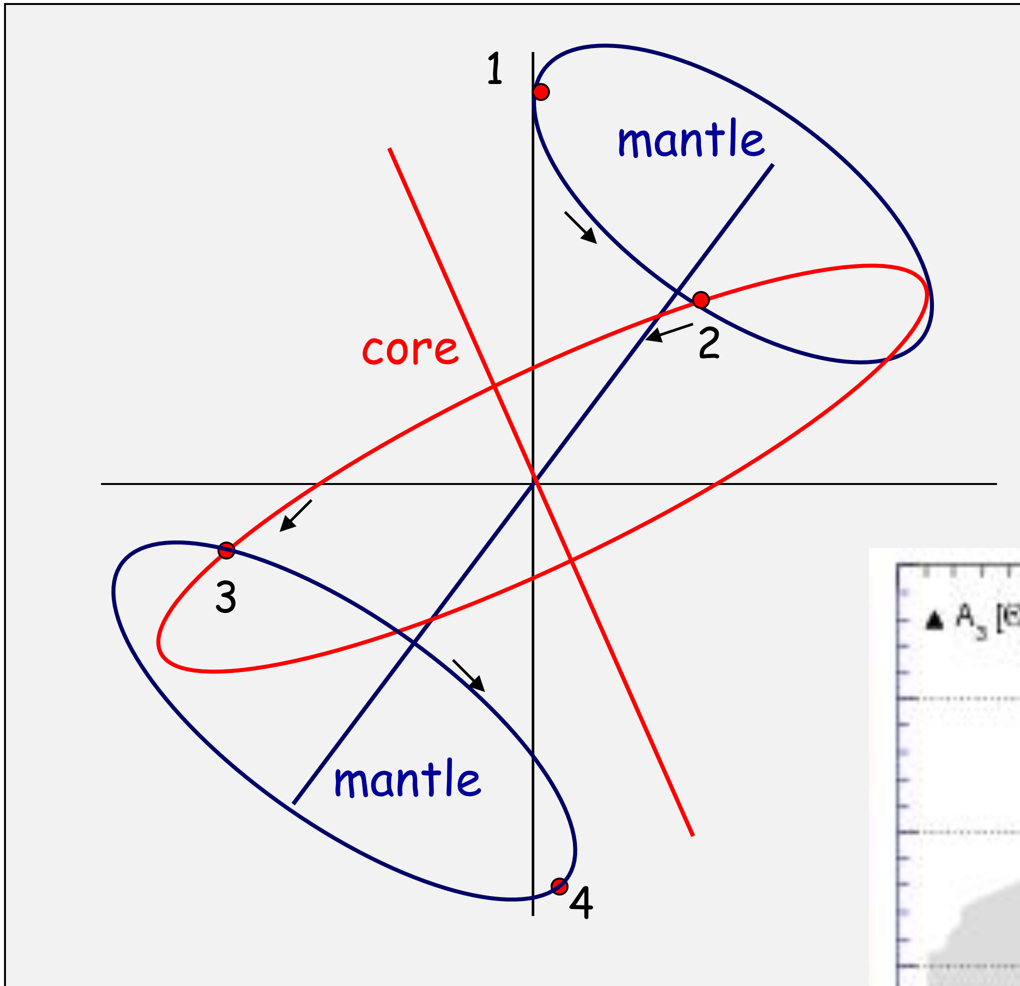
``Castle wall profile"



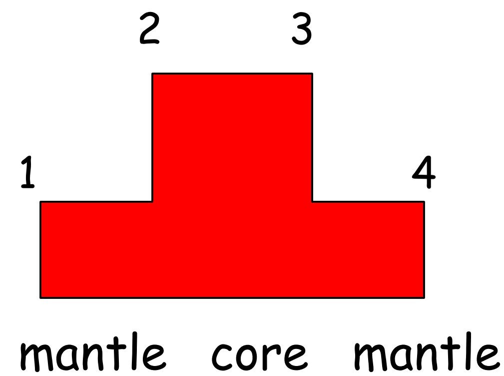
``Castle wall profile"

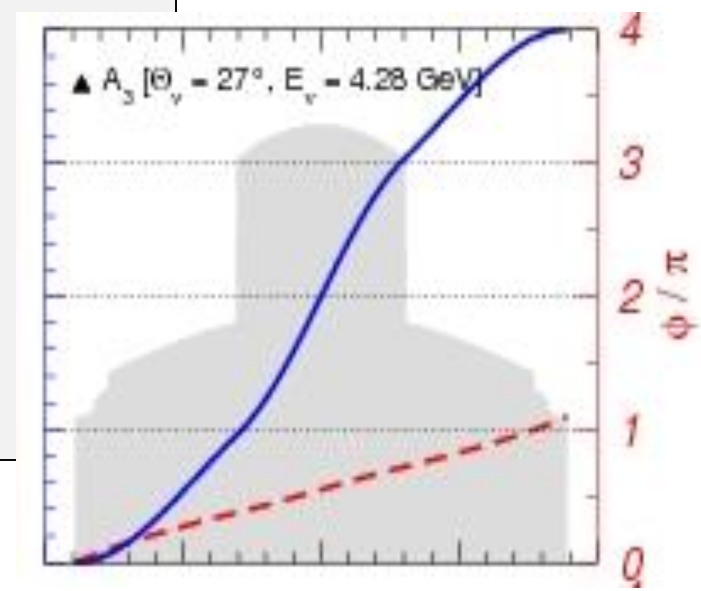


Parametric enhancement in the Earth, 1-3 mode



In the Earth





Fuzzy dark matter and parametric resonance

Ultra-light scalar DM

A. Berlin, 1608.01307 [hep-ph]

$$\phi(t, x) \sim \frac{\sqrt{2\rho(x)}}{m_\phi} \cos(m_\phi t)$$

Couples to neutrinos: $g_\phi \phi \nu_i \nu_j + \ldots$

give contribution to neutrino mass and modifies mixing

$\delta m(t) = g_\phi \phi(t)$ $\Delta\theta_m(t) = g_\phi \phi(t) / \Delta m_{ij}$

Neutrinos propagating in this field will experience variation of mixing with frequency given by m_ϕ → mass states oscillate

For $m_\phi = 10^{-22}$ eV, the modulation length $l_{mod} = 2\pi/m_\phi = 10^{17}$ cm

Parametric resonance: $l_{mod} = l_\nu$

For solar mass splitting $E_{res} = 3 \times 10^3$ PeV $(10^{-22}\ \mathrm{eV}/m_\phi)$

$E_{res} = 3$ PeV for $m_\phi = 10^{-19}$ eV

For $m_\phi = 10^{-10}$ eV, $l_{mod} \sim 10^5$ cm →

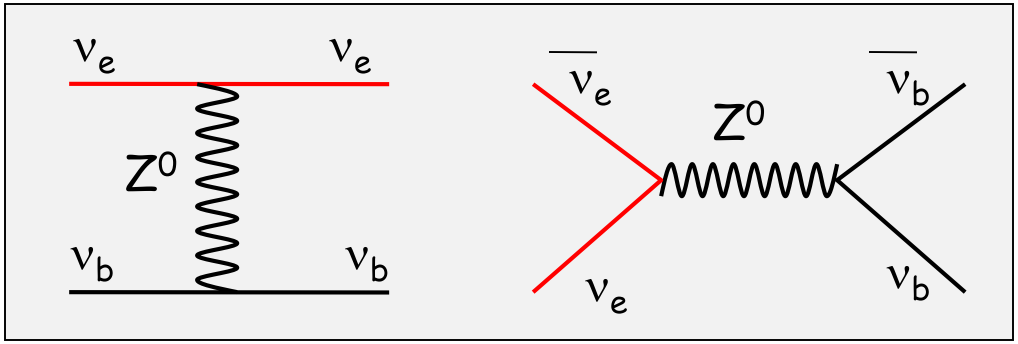
Effects in lab. Experiments, KamLAND, JUNO

M. Losada et al, 2205.09769 [hep-ph]

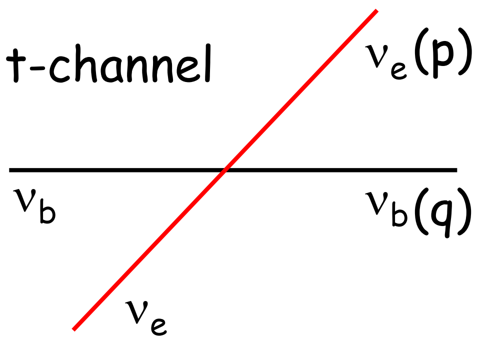
4. $\nu\nu$- scattering and collective transformations

Another possible
manifestation of
parametric resonance

$\nu\nu$-scattering



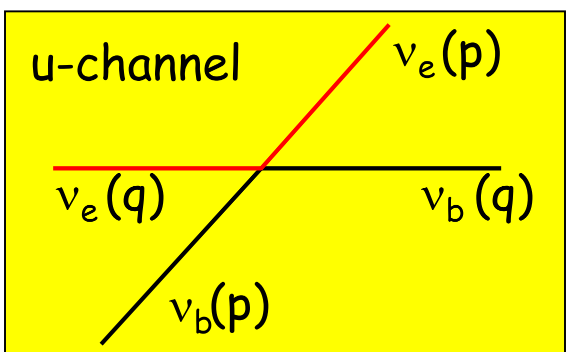
Refraction in neutrino gases



elastic forward scattering
→ coherent

$$V = \sqrt{2}\, G_F \,(1 - v_e\, v_b\,)\, n_b$$

velocities



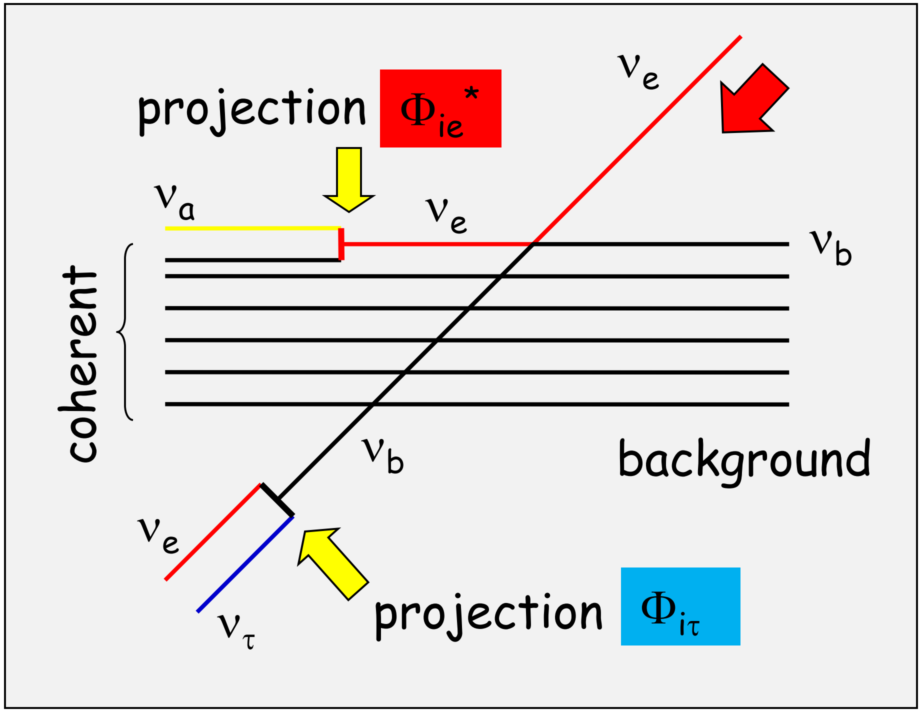
J. Pantaleone

Momentum exchange = flavor exchange
→ flavor mixing

Can it be coherent?

Coherence of flavor exchange

J. Pantaleone
S. Samuel
V.A. Kostelecky



Coherence if the background is a mixture of flavor states:

for ith particle of background

$$|\nu_{ib}\rangle = \Phi_{ie}|\nu_e\rangle + \Phi_{i\tau}|\nu_\tau\rangle$$

Φ_{ie} - ($\Phi_{i\tau}$ -) amplitude to find ν_e (ν_τ) in ith bkgr. neutrino

Inverting

$$|\nu_e\rangle = \Phi_{ie}^*|\nu_b\rangle + \Phi_{i\tau}^*|\nu_a\rangle$$

transition $\nu_e + \nu_b \rightarrow \nu_\tau + \nu_b$ with amplitude $\sim \Phi_{ie}^*\Phi_{i\tau}$

and unchanged background

→ summation over background neutrinos is coherent

This generates flavor non - diagonal potential $V_{e\tau} \sim \Sigma_i\, \Phi_{ie}^*\Phi_{i\tau}$

also diagonal one

Neutrino term in the Hamiltonian

Contribution to the Hamiltonian in the flavor basis

$$H_{\nu\nu} = \sqrt{2}\, G_F \Sigma_i (1 - v_e v_b) \begin{pmatrix} |\Phi_{ie}|^2 & \Phi_{ie}^{*}\Phi_{i\tau} \\ \Phi_{ie}\Phi_{i\tau}^{*} & |\Phi_{i\tau}|^2 \end{pmatrix} \qquad \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix}$$

where $\Phi_{ie} = \sqrt{P_{be}}$ $\quad \Phi_{i\tau} = \sqrt{P_{b\tau}}$

The Hamiltonian in symmetric form:

$$H_{\nu\nu} = \tfrac{1}{2} \begin{pmatrix} V_\nu & 2\overline{V}_\nu e^{i\phi} \\ 2\overline{V}_\nu e^{-i\phi} & - V_\nu \end{pmatrix}$$

where

$V_\nu \sim \sqrt{2}\, G_F n(1 - v_e v_b)(P_{be} - P_{b\tau})$ $\qquad \overline{V}_\nu \sim \sqrt{2}\, G_F n(1 - v_e v_b) \sqrt{P_{be}P_{b\tau}}$

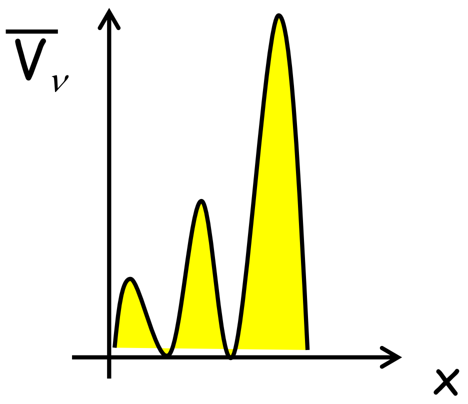
The effective coupling constants in V include probabilities

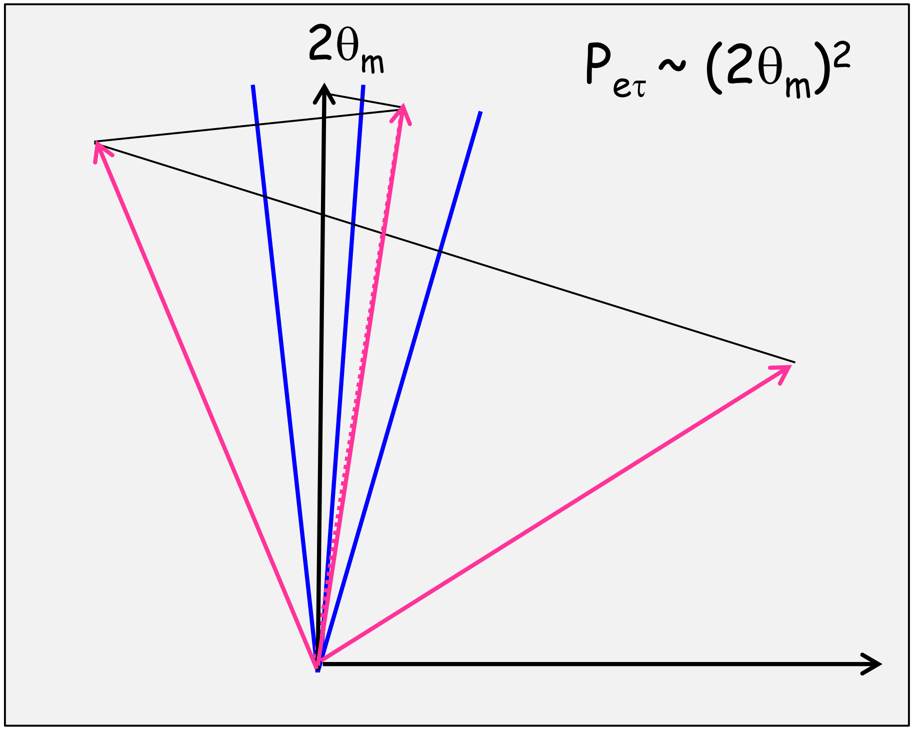
Instabilities and fast transition in the flavor field

Exponential grow of the transition probabilities

In the Hamiltonian depend on probabilities which have oscillatory dependence

Parametric enhancement induced by modulations of the neutrino potentials with growing amplitude





$\Delta\theta_m / \Delta t \sim \theta_m$

The cone angle and transition probability increase exponentially

Backup

Oscillation length in matter

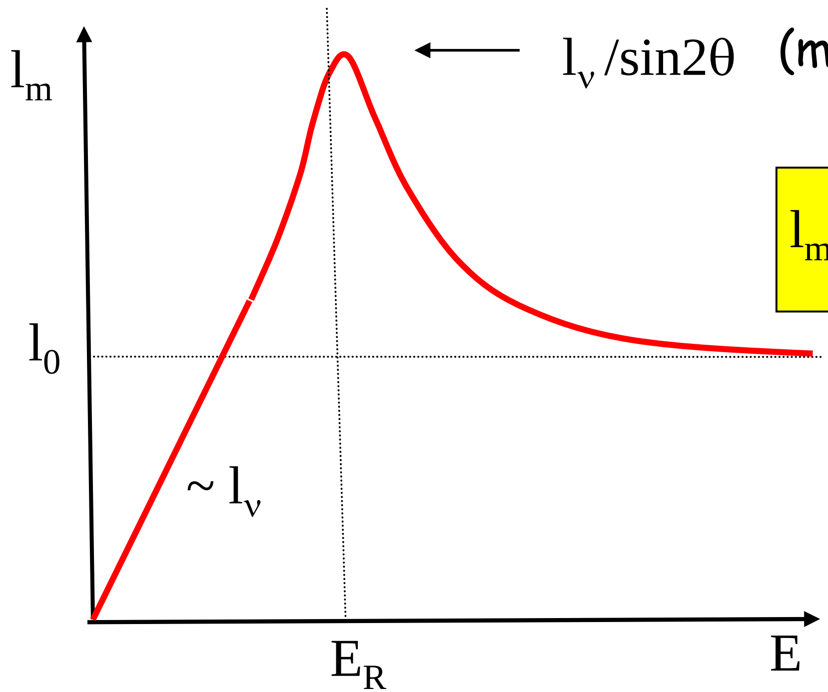
Oscillation length in vacuum

$$l_\nu = \frac{4\pi E}{\Delta m^2}$$

Refraction length

$$l_0 = \frac{2\pi}{\sqrt{2}\, G_F n_e}$$

- determines the phase produced by interaction with matter



(maximum at $l_\nu = l_0/\cos 2\theta$)

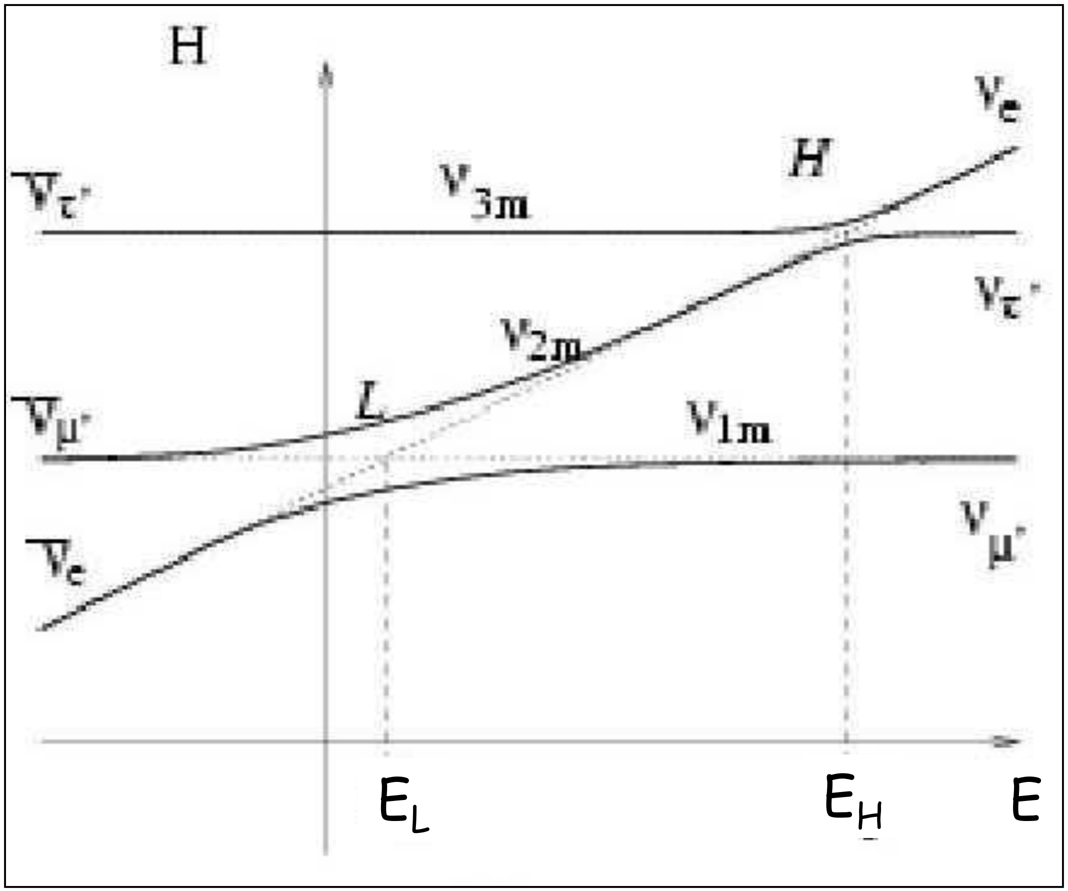
$$l_m = \frac{2\pi}{H_{2m} - H_{1m}}$$

shifts with respect resonance energy:

$$l_\nu(E_R) = l_0 \cos 2\theta$$

converges to the refraction length

Level crossings



Normal mass ordering

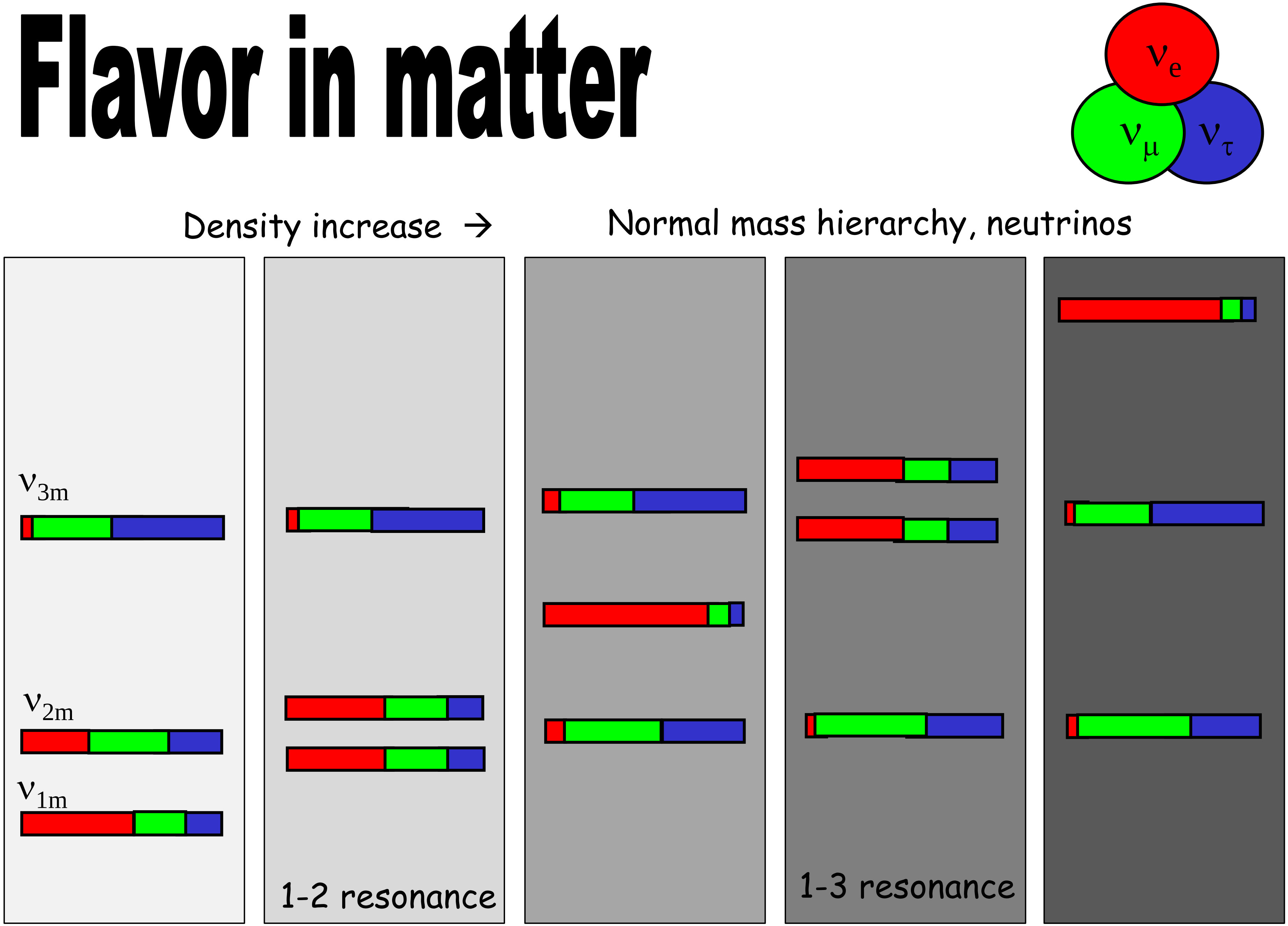
0.1 GeV

6 GeV

Resonance region

High energy range

Flavor in matter



Parametric enhancement in the Earth,1-2 mode

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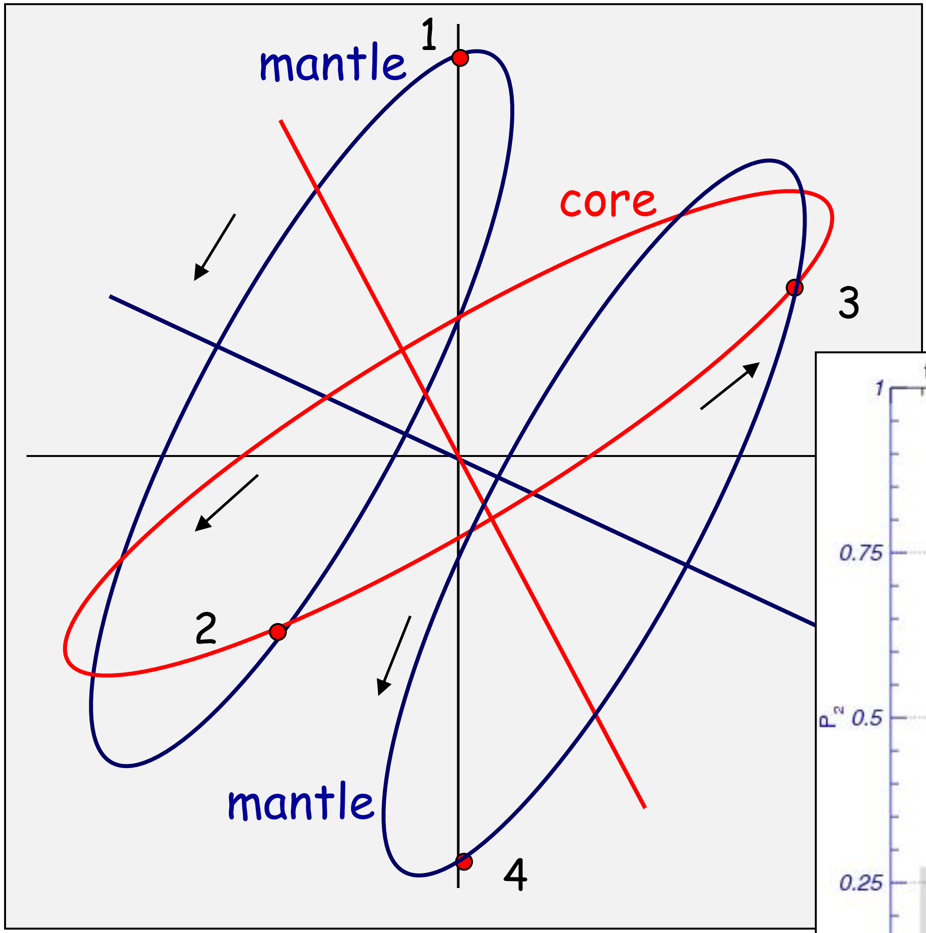
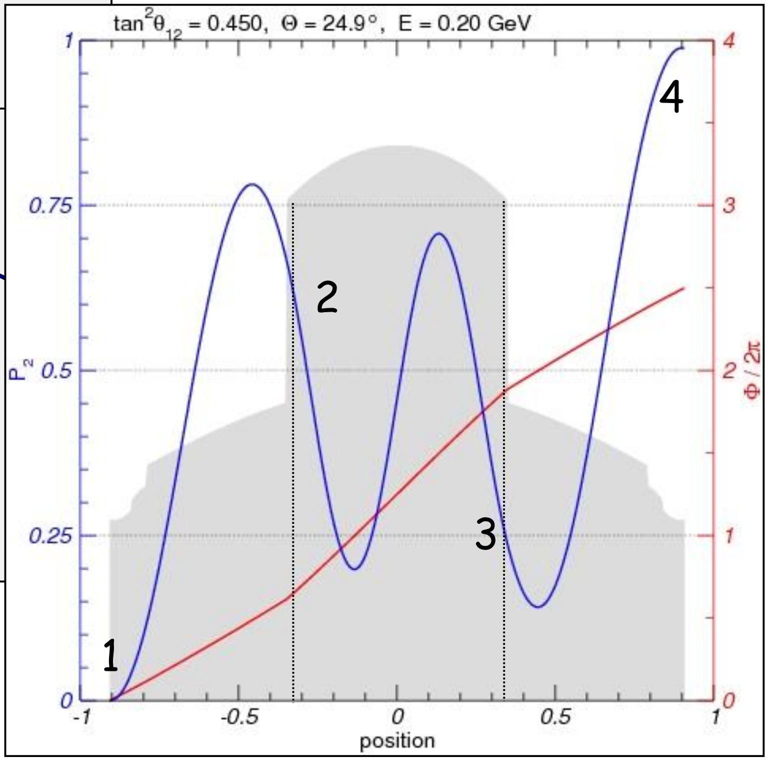




Total Hamiltonian

$$H = \frac{1}{2}\begin{pmatrix} -\cos 2\theta\, \omega_p + V_e + V_\nu & \sin 2\theta\, \omega_p + 2\bar{V}_\nu e^{i\phi} \\ \sin 2\theta\, \omega_p + 2\bar{V}_\nu e^{-i\phi} & \cos 2\theta\, \omega_p - V_e - V_\nu \end{pmatrix} \qquad \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix}$$

includes the vacuum contribution: $\omega_p = \Delta m^2/2E$ and usual matter potential V_e

Neutrino potentials

$$V_\nu \sim V_\nu^0 (1 - P^B_{e\tau})$$

$$\bar{V}_\nu \sim V_\nu^0 \sqrt{P^B_{e\tau}(1 - P^B_{e\tau})}$$

if ν_e is produced

$\phi = Arg\,[\Phi_e \Phi_\tau^*]$

$P^B_{e\tau}(x)$ – effective transition probability of the background neutrinos

In the central parts of collapsing star $V_e \gg V_\nu \gg \omega$

Evolution equation

Ensemble of neutrino polarization vectors $\mathbf{P}_\omega$

Negative frequencies for antineutrinos

$$d_t \mathbf{P}_\omega = (-\omega \mathbf{B} + \lambda \mathbf{L} + \mu \mathbf{P}) \times \mathbf{P}_\omega$$

Vacuum mixing term

$\mathbf{B} = (\sin 2\theta\,,\ 0,\ \cos 2\theta)$

$\omega = \Delta m^2 / 2E$

Usual matter potential

$\mathbf{L} = (0,\ 0,\ 1)$

$\lambda = V = \sqrt{2}\, G_F\, n_e$

Collective vector

$$\mathbf{P} = \int_{-\text{inf}}^{+\text{inf}} d\omega\ \mathbf{P}_\omega$$

$\mu = \sqrt{2}\, G_F n_\nu (1 - \cos \theta_{\nu\nu})$

The term describes collective effects

Effective theory of collective oscillations

Difficult to solve although

Some results can be seen from general form of the evolution equation

effects which can lead to strong flavor transformations
Conditions for these effects

problems of realizations of these conditions in realistic supernova

Total Hamiltonian and potentials

$$H = \frac{1}{2}\begin{pmatrix} -\cos 2\theta\,\omega_p + V_e + V_\nu & \sin 2\theta\,\omega_p + 2\bar{V}_\nu e^{i\phi} \\ \sin 2\theta\,\omega_p + 2\bar{V}_\nu e^{-i\phi} & \cos 2\theta\,\omega_p - V_e - V_\nu \end{pmatrix} \quad \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix}$$

Potentials

$$V_\nu \sim V_\nu^0 (1 - P^B_{e\tau})$$

$$\bar{V}_\nu \sim V_\nu^0 \sqrt{P^B_{e\tau}(1 - P^B_{e\tau})}$$

$P^B_{e\tau} = P_{e\tau}$

non-linearity

$P^B_{e\tau}(x)$ – effective transition probability of the background neutrinos

$$V_e \gg V_\nu \gg \omega$$

$H^{diag} \sim V_e$ $\quad H^{non\text{-}diag} \sim V_\nu^0 \sqrt{P^b_{e\tau}} < V_\nu^0$ $\quad \phi \sim \int dt\, \Delta H$

$\Delta H \sim V_e$ $\quad d\phi/dt \sim V_e$

if $\omega \ll V_\nu$, H depends on potentials only – evolution of neutrinos and antineutrinos is the same → bi-polar oscillations

Properties of potentials

Potentials are integrals of oscillation amplitudes which have oscillatory dependence on time.

Therefore potentials are also expected to be oscillatory functions of time determined by intrinsic frequencies of the system

$$V_e\ ,\ V_\nu,\ \ \omega_p,\ \ \omega_k$$

Furthermore, there is the hierarchy of frequencies

$$V_e > V_\nu \gg \omega$$

Parametric enhancement

Parametric resonance if the frequency of modulations of potentials coincides with eigenfrequency of the probe neutrino

Two effects of enhancement

Phase velocity cancellation

Rotation of the fields that eliminates the phase from the off-diagonal terms leads to appearance of phase velocity in the diagonal terms

$$V^r(t) = V_e + V_\nu - \cos 2\theta\, \omega_p - d\phi/dt$$

if $d\phi/dt \sim V_e + V_\nu$ strong cancellation → matter suppression is removed

Oscillations with maximal depth and frequency $1/V_\nu$

Parametric enhancement

V_ν and $\overline{V}_\nu$ - periodic functions

Parametric resonance if the frequency of modulations of potentials coincides with eigenfrequency of the probe neutrino

Mixing in matter – dynamical variable

S.P. Mikheyev, A.Y.S. 1985

Since

$$H(n_e, E) = H_0(E) + V(n_e)$$

the mixing matrix in matter which diagonalizes $H(n_e, E)$

$$U^m = U^m(n_e, E)$$

depends on energy and density → becomes dynamical variable in non-uniform medium in contrast to vacuum mixing which is constant

Inverting relation:

$$\nu_m = U^{m+}(n_e, E)\, \nu_f$$

which means that flavor composition of eigenstates depends on n_e and E

Mixing in matter

in vacuum:

H_0

$\nu_1 \quad \nu_2$ $\quad \nu_{mass}$

Eigenstates of H_0 in vacuum

ν_f $\quad \nu_e \quad \nu_\mu$

θ $\quad \theta_m$

in matter:

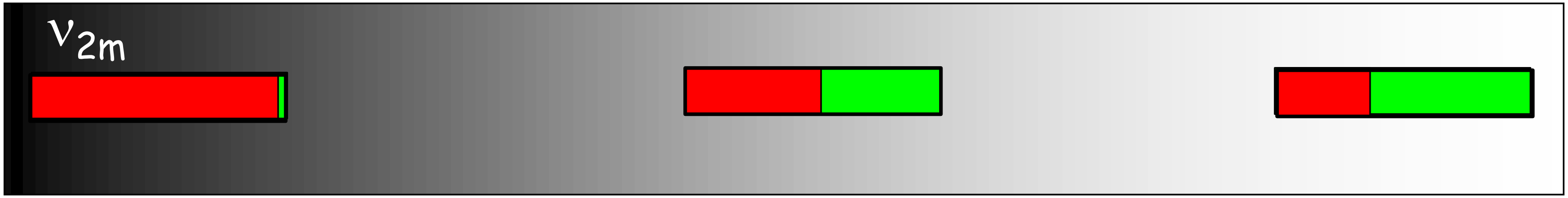
$H(n_e, E) = H_0 + V$

ν_H $\quad \nu_{1m} \quad \nu_{2m}$

Eigenstates of H in medium

θ_m depends on n_e, E

Mixing angle determines flavor content of eigenstates of propagation



High density
mixing suppressed

Resonance: $l_\nu = l_0 \cos 2\theta$
- maximal mixing

Low density
Vacuum mixing

Hamiltonian for flavor states in matter

rep

In the flavor basis $\nu_f = (\nu_e, \nu_\tau)^T$

$$H_{tot} = \frac{M M^+}{2E} + V(t)$$

$$M M^+ = U M_{diag}^2 U^+$$

$$M_{diag}^2 = \mathrm{diag}(m_1^2, m_2^2)$$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$H_{tot} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta + \xi & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$\xi = \frac{4 V_e E}{\Delta m^2} \qquad V_e = \sqrt{2}\, G_F n_e$$

Evolution equation for eigenstates

**

Explicitly for 2ν mixing

$$i \frac{d\nu_f}{dt} = H_{tot}\, \nu_f$$

$$H_{tot} = H_{tot}(n_e(t))$$

$$\nu_f = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \qquad \nu_m = \begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix}$$

Inserting $\nu_f = U(\theta_m)\, \nu_m$ $\quad \theta_m = \theta_m(n_e(t))$

$$i \frac{d}{dt} \begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix} = \begin{pmatrix} 0 & i \frac{d\theta_m}{dt} \\ -i \frac{d\theta_m}{dt} & H_{2m} - H_{1m} \end{pmatrix} \begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix}$$

off-diagonal terms imply transitios

$\nu_{1m} \Leftrightarrow \nu_{2m}$

if $\left| \frac{d\theta_m}{dt} \right| \ll H_{2m} - H_{1m}$

off-diagonal elements can be neglected
no transitions between eigenstates
propagate independently

Adiabaticity

Adiabaticity condition $\left|\frac{d\theta_m}{dt}\right| \ll H_{2m} - H_{1m}$

External conditions (density) change slowly the system has time to adjust them

transitions between the neutrino eigenstates can be neglected $\nu_{1m} \not\leftrightarrow \nu_{2m}$ → The eigenstates propagate independently

Shape factors of the eigenstates do not change

Crucial in the resonance layer:
- the mixing changes fast
- level splitting is minimal

Adiabaticity condition $\Delta r_R > l_R$ if vacuum mixing is small

$\Delta r_R = n_R \tan 2\theta / (dn/dx)_R$ $l_R = l_\nu / \sin 2\theta$

width of the resonance layer

oscillation length in resonance

Adiabatic parameter

$$\gamma = \frac{\left| \frac{d\theta_m}{dt} \right|}{H_{2m} - H_{1m}}$$

Adiabaticity condition:
$\gamma << 1$

most crucial in the resonance where
the mixing angle in matter changes fast

$$\gamma_R = \frac{l_R}{2\pi \, \Delta r_R}$$

$\Delta r_R = h_n \tan 2\theta$ is the width of the resonance layer

$h_n = \frac{n}{dn/dx}$ is the scale of density change

$l_R = l_\nu / \sin 2\theta$ is the oscillation length in resonance

Explicitly:

$$\gamma_R = \frac{4\pi E \cos 2\theta}{\Delta m^2 \sin^2 2\theta \; h_n}$$

Spatial picture

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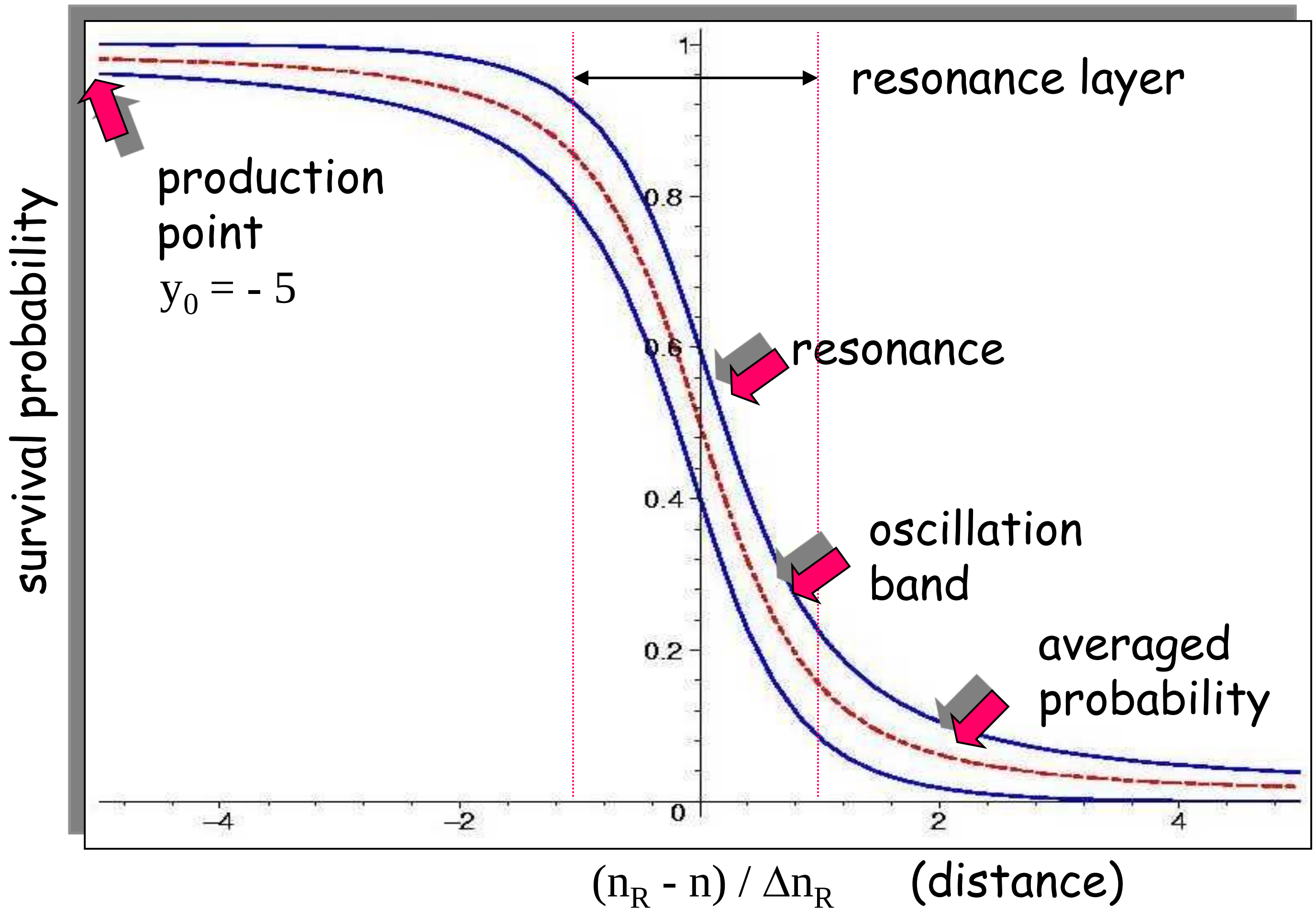
The picture is universal in terms of variable $y = (n_R - n) / \Delta n_R$
no explicit dependence on oscillation parameters, density distribution, etc.
only initial value y_0 matters



Oscillations vs. adiabatic conversion **

Different degrees of freedom

Oscillations

Vacuum or uniform medium with constant parameters

Phase difference increase between the eigenstates

$\phi(t)$

Mixing does not change

$\theta_m(E)$

Adiabatic conversion

Non-uniform medium or/and medium with varying in time parameters

Change of mixing in medium = change of flavor of the eigenstates

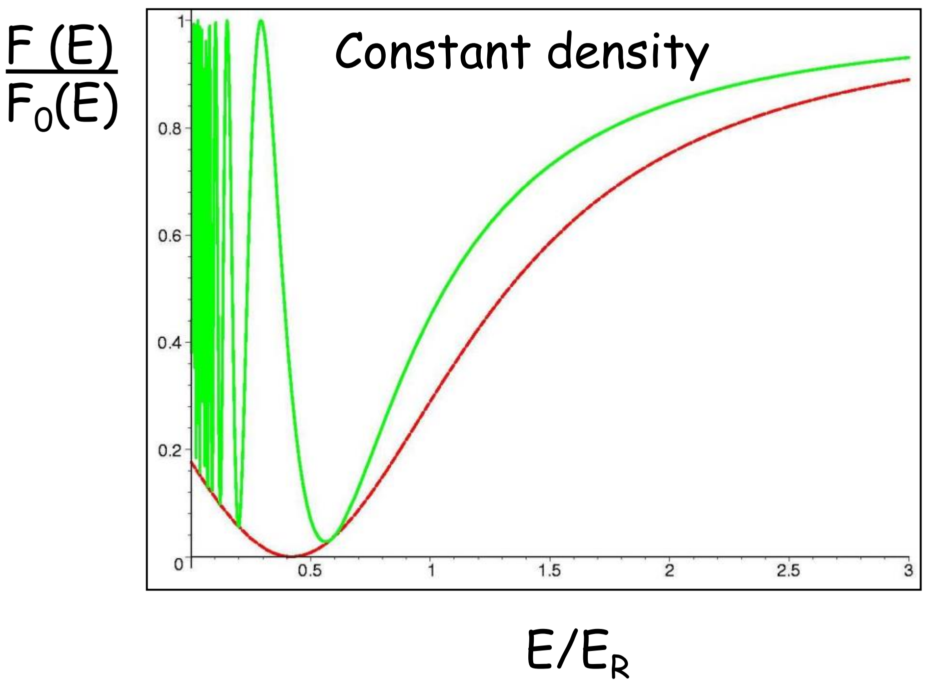
$\theta_m(t)$

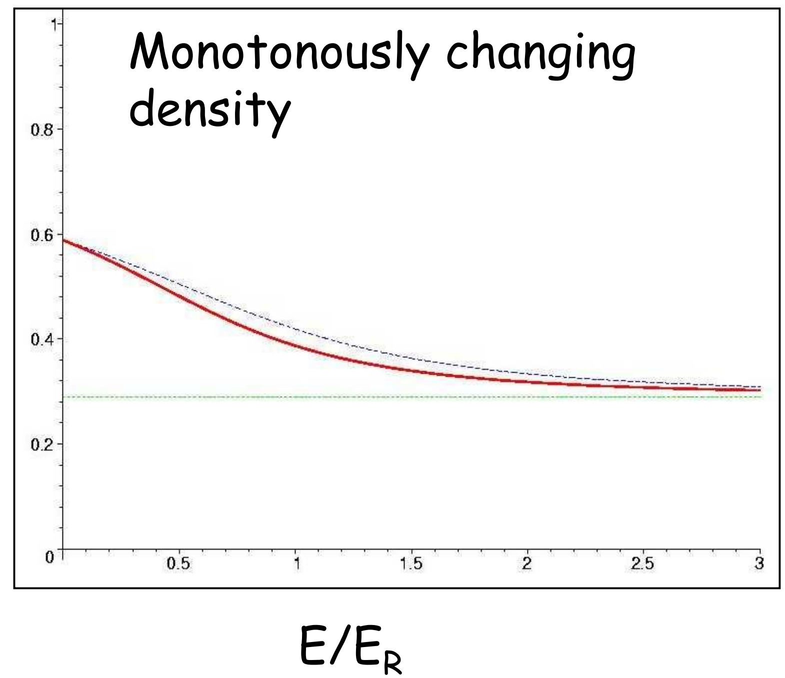
Phase is irrelevant

In non-uniform medium: interplay of both processes

Resonance oscillations vs. adiabatic conversion **

Passing through the matter filter





Parametric enhancement in the Earth,1-2 mode

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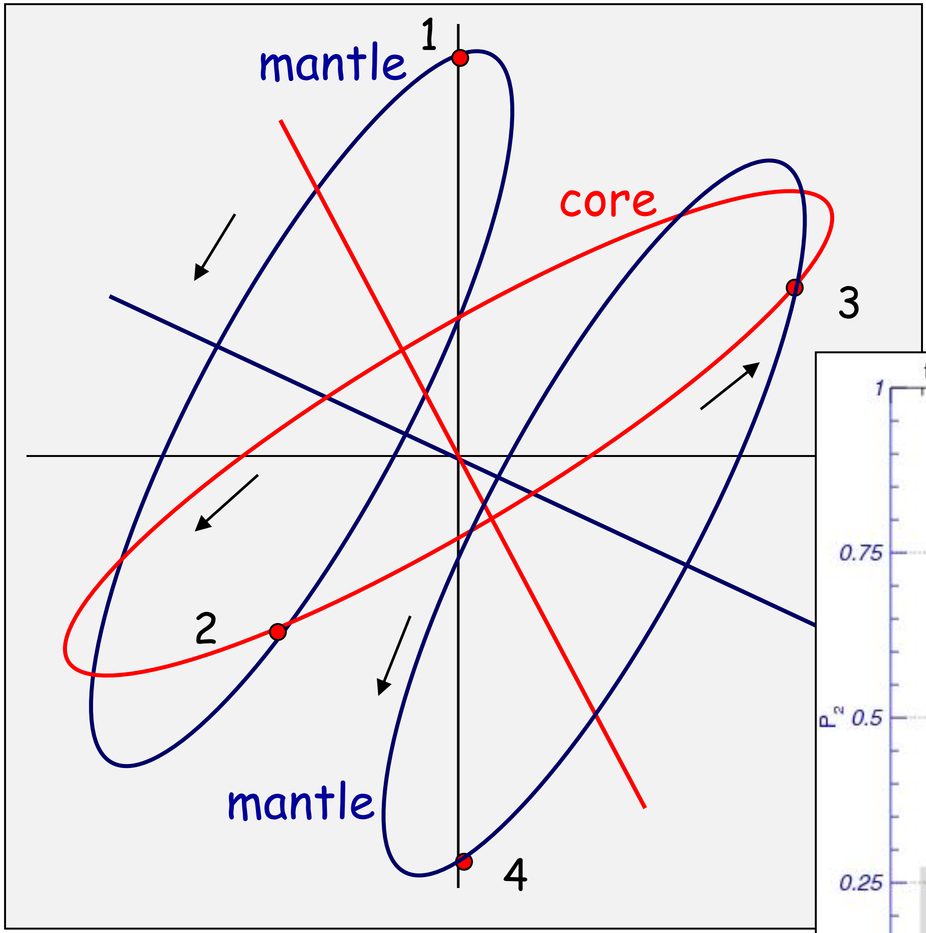
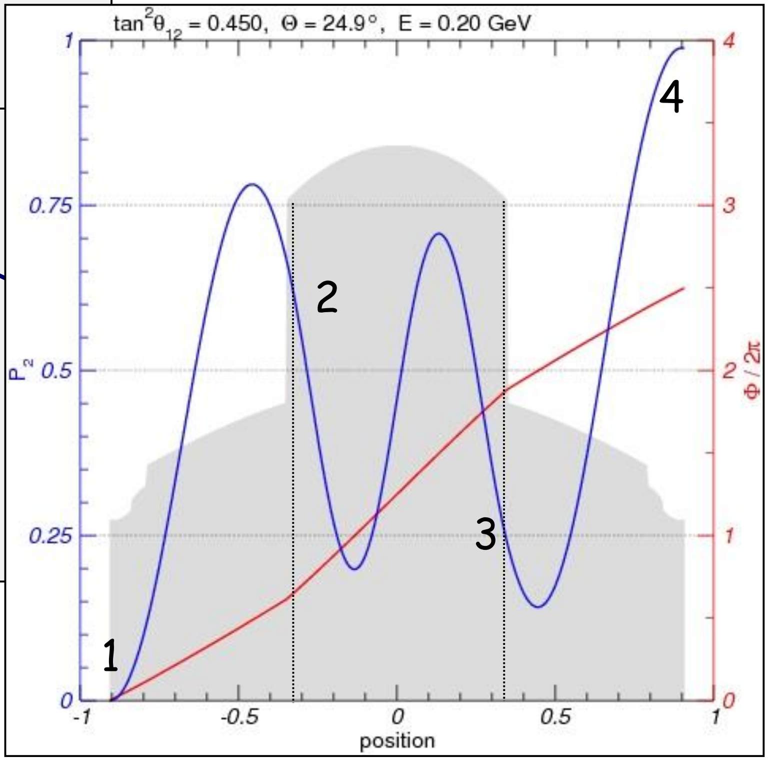




Conditions for strong transformations

1. Resonance oscillations

$V^r \ll V'$ ⇨ $V_e + V_\nu + d\phi'/dt - \cos 2\theta\, \omega_p \sim 0$

or $d\phi'/dt \sim - V_e - V_\nu$

The system oscillate with maximal depth and frequency ~ V'
If there is no significant modulations of the non-diagonal element

2. Adiabatic conversion

Performing series of transformations of fields - exclude fast time variations in V^r and V'

In new frame $\tilde{V}^r$ and $\tilde{V}'$ may satisfy adiabatic condition →
strong transition if $\tilde{V}^r$ changes from $\tilde{V}^r \gg \tilde{V}'$ to $\tilde{V}^r \ll \tilde{V}'$

... continued

**

3. Parametric enhancement, resonance

Potentials are modulated by periodic functions, so that the mixing angle in medium $\tan 2\theta_m = - V' / V^r$ varies with a period T_θ

Parametric enhancement if the frequency of modulations coincides with eigenfrequency of the system $1/T_p$

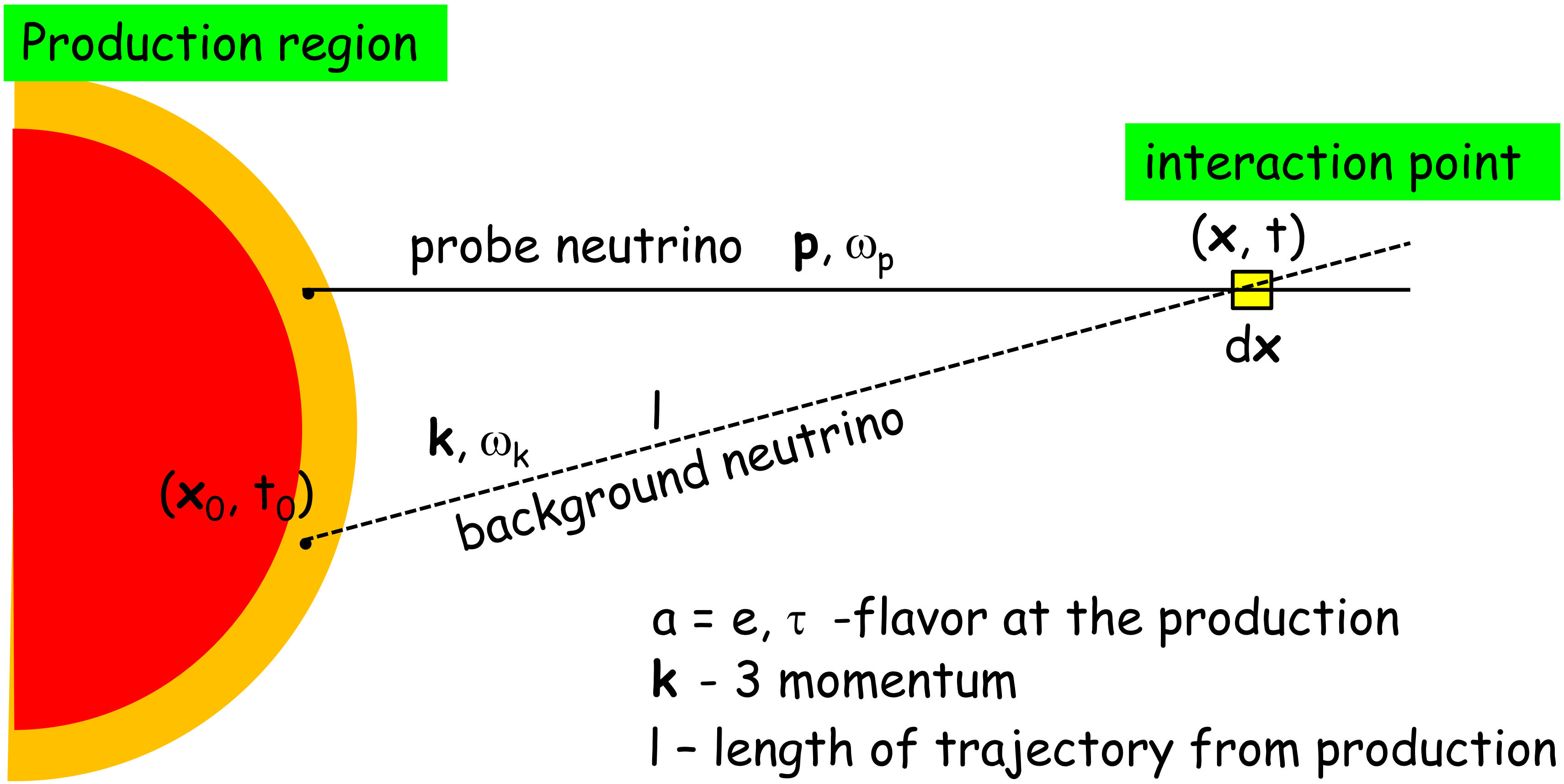
$$T_\theta = T_p$$

$$T_p = \frac{2\pi}{\sqrt{\langle V^r \rangle^2 + \langle V' \rangle^2}}$$

$\langle V^r \rangle$, $\langle V' \rangle$ - potentials averaged over modulations

Large transition probability develops over many periods

Summation over background neutrinos



Production point - the point of last inelastic collision

$a = e, \tau$ -flavor at the production

k - 3 momentum

l – length of trajectory from production $(\mathbf{x}_0, t_0)$ to interaction point $(\mathbf{x}, t)$

→ $(a, l, \mathbf{k})$ characterize a given mode

$(\mathbf{x}, t)$, **k** and l determine $(\mathbf{x}_0, t_0)$

Total Hamiltonian and potentials

After integration

$$H = \frac{1}{2}\begin{pmatrix} -\cos 2\theta\, \omega_p + V_e + V_\nu & \sin 2\theta\, \omega_p + 2\bar{V}_\nu e^{i\phi} \\ \sin 2\theta\, \omega_p + 2\bar{V}_\nu e^{-i\phi} & \cos 2\theta\, \omega_p - V_e - V_\nu \end{pmatrix} \qquad \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix}$$

Potentials

$$V_\nu = \int d\mathbf{k} \int dl \;\; [V_\nu^e(\mathbf{k}, l) - V_\nu^\tau(\mathbf{k}, l)]\,(1 - P_{e\tau}(\mathbf{k}, l))$$

$$\bar{V}_\nu e^{i\phi} = \int d\mathbf{k} \int dl \;\; [V_\nu^e(\mathbf{k}, l) - V_\nu^\tau(\mathbf{k}, l)]\, e^{i\phi(\mathbf{k}, l)} \sqrt{P_{e\tau}(\mathbf{k}, l)\,(1 - P_{e\tau}(\mathbf{k}, l))}$$

where

$$V_\nu^a(\mathbf{k}, l) = \sqrt{2}\, G_F\, n_\nu^a(\mathbf{k}, l)\,(1 - \mathbf{v}_p \mathbf{v}_k) \qquad a = e, \tau$$

$n_\nu^a(\mathbf{k}, l)$ - number density of neutrinos emitted from $(\mathbf{x}_0, t_0)$ and arriving at the point $(\mathbf{x}, t)$

Removing the phase

Off-diagonal term: introduce real V' and the phase ϕ' as

$$\sin 2\theta\, \omega_p + 2\overline{V}_\nu e^{i\phi} = V' e^{-i\phi'} \qquad (*)$$

Transformation of the fields

$$\psi = U\psi' \qquad U = \mathrm{diag}\,(e^{0.5\, i\phi'},\ e^{-0.5 i\phi'})$$

it does not change flavor probabilities.

Hamiltonian for ψ'

$$H = \frac{1}{2}\begin{pmatrix} V^r(t) & V'(t) \\ V'(t) & -V^r(t) \end{pmatrix}$$

$$V^r(t) = V_e + V_\nu - \cos 2\theta\, \omega_p + d\phi'/dt \qquad V'(t) \text{ is def. in } (*)$$

Probe neutrino propagation in external neutrino potentials (as in the case of NSI but) with non-trivial time dependence

The problem

Evolution equation for the probe particle

$$i \frac{d\,\Phi_p}{d\,t} = H\,(\Phi_{ib}^{x})\,\Phi_p$$

$x = ct$ – point along the ν_p trajectory

Total Hamiltonian depends on the wave functions of all background neutrinos Φ_{ib}^{x} which cross the probe neutrino trajectory in a point **x**

To find Φ_{ib}^{x} one needs to solve the corresponding evolution equation for each ν_{ib}

$$i \frac{d\,\Phi_{ib}}{d\,t} = H\,(\Phi_{jb}^{y})\,\Phi_{ib}$$

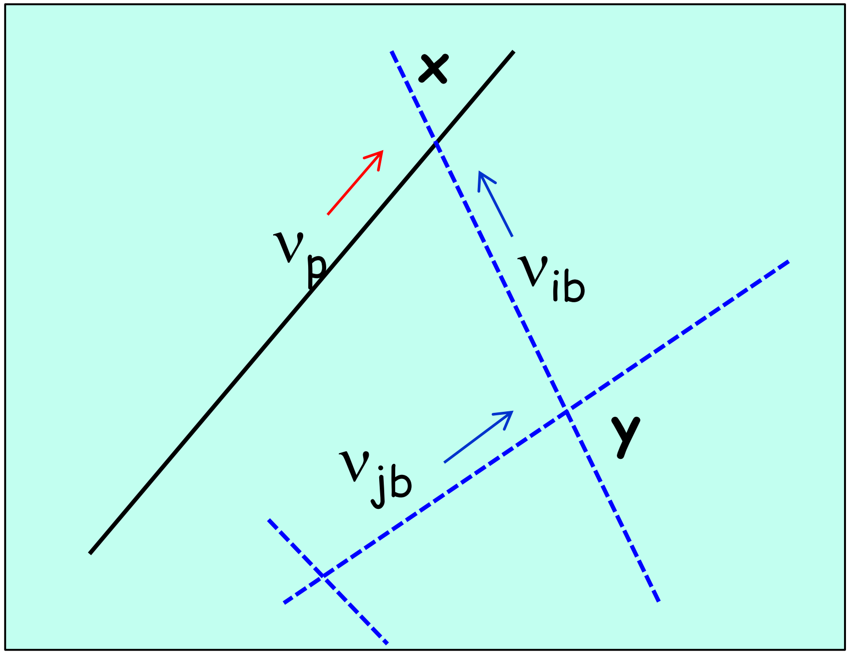
$y = ct$ along the ν_{ib} trajectory

Here H depends on the wave functions of all background neutrinos Φ_{jb}^{y} which cross the ν_{ib} trajectory in a point **y**

The equation should be integrated over **y** from the ν_{ib} production point to **x**

→ Huge number of coupled equations

The problem



Stationary case: pattern does not depend on time

Modes involved are characterized by ($\mathbf{x}_0$, $\mathbf{k}$, a)
$a = e, \tau$ -flavor at production
$\mathbf{x}_0$ - production point
$\mathbf{k}$ - momentum

Specific limits of integration for each point

Modes with the same set ($\mathbf{x}_0$, $\mathbf{k}$, a) evolve in the same way

Discretize parameters – numerical solutions

Potentials: time dependence

Combining different terms in the Hamiltonian (in rotating frame)

$$H = \frac{1}{2}\begin{pmatrix} V^r(t) & V'(t) \\ V'(t) & -V^r(t) \end{pmatrix}$$

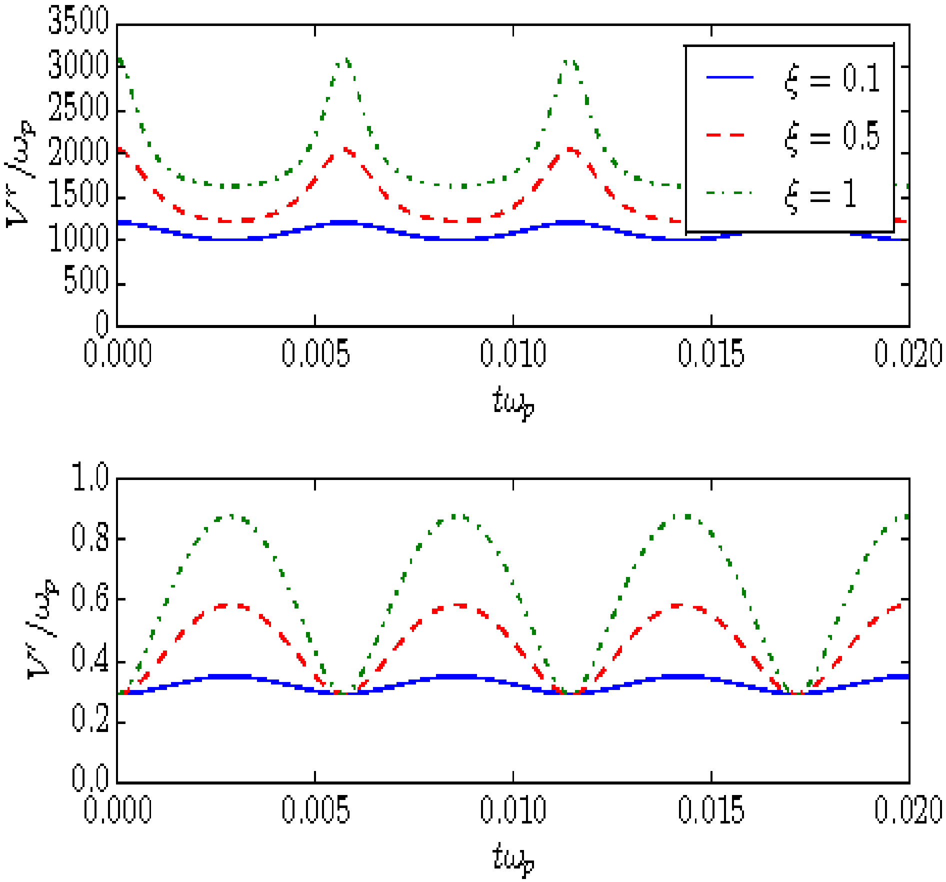
including usual matter potential V_e which dominates

$$V_e = 10^3\,\omega_k$$

and vacuum terms

Neutrino to matter potential ratio

$$\xi = \frac{V_\nu^0}{V_e}$$



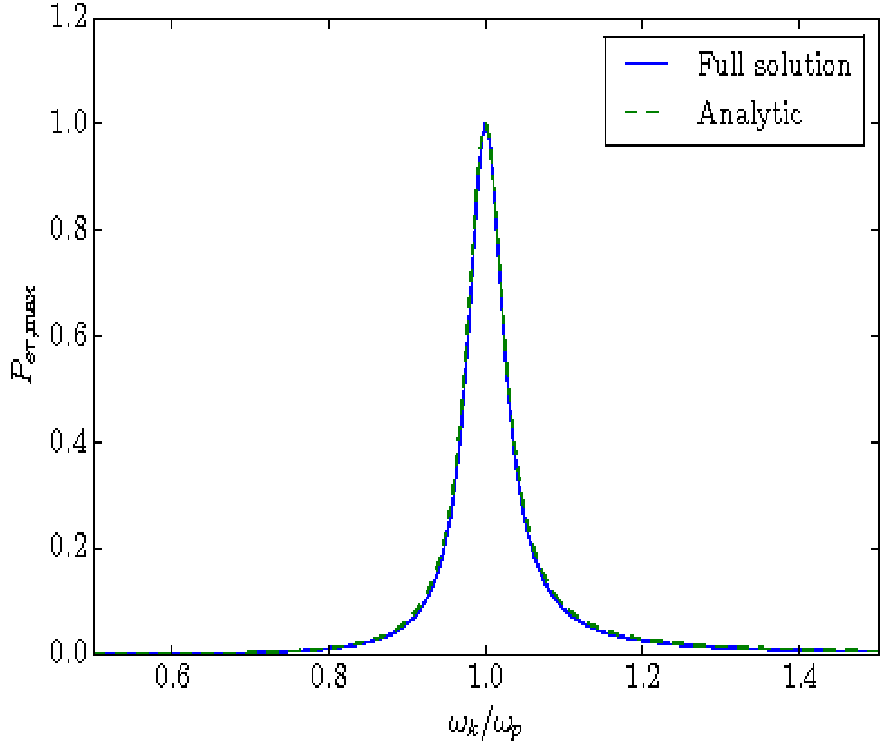
V_r and V' as functions of the time for different values of ξ. $A_\beta = 1.1001$, $\omega_p = \omega_k$

Parametric resonance

Dependence of the depth of parametric oscillations on ω_k / ω_p

$A_\beta = 1.1001$, $V_\nu^0 = 100\ \omega_p$

$V_\nu^0 = 100\ \omega_p$



Parametric resonance condition:

$$\underbrace{-\cos 2\theta\ \omega_p + V_e + V_\nu^0}\ = \underbrace{A_\beta\, \Delta_m}$$

Frequency of oscillations of the probe neutrino determined by the averaged density

Frequency of modulations of the potentials

Width of the resonance

$\Gamma / \omega_k = \sin 2\theta\ \xi\ (1 + A_\beta)$

proportional to ξ

Inverse problem:

Find potentials and their time dependences or general conditions for potentials which can lead to effects found in certain simplified models

Synchronized oscillations

Fast flavor transitions

As parametric effect with increasing amplitude of periodic modulations

Bi-polar oscillations

As parametric effect for negligible ω → transition is the same for neutrinos and antineutrinos

Spectral splits

Check how realistic are these conditions in realistic supernova

WP's and non-adiabatic evolution

Partially ionized atoms as the electron density perturbations

Number density profile of electrons in atom (O, C, He) is non adiabatic

M. Kusakabe
2109.11942 [hep- ph]

Interplay of non-adiabatic evolution and separation (relative shift) of the WP's leads to new effects: additional averaging of oscillations

Applications to Supernova neutrinos

No new effects without WP separation and adiabatic evolution

No new effects for very sharp (step-like) density profile

Evolution of WP's

WP's are formed at the production (at boundaries)

$\psi(t\ x) = \int dp\ f(p)\ \phi_p(t\ x)$ $\phi_p(t\ x)$ - plane waves

If there is no absorption or p-dependent interactions, f(p) does not change in the process of evolution

Evolution equation $i d\psi/dt - H\,\psi = 0$, insert $\psi(t\ x)$:

$$\int dp\ f(p)\ [i d\phi_p/dt - H\phi_p\ ] = 0$$

Superposition principle and linearity of evolution equation → solve eq for ϕ_p, then integrate over p (which takes care about WP nature)
No effects predicted in *2109.11942 [hep- ph]*

In t-x space WP can change form in the course of evolution, but integrated over time result coincides with result in E-p rep.

Y. P. Porto-Silva , A Y S
2103.10149 [hep-ph]

$\nu\nu$ - scattering → $H = H(\phi_p)$ – non-linear equation ?