NEUTRINOS IN HOT AND DENSE MEDIA

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Outline

- How to define a hot and dense medium.
- Development of statistical quantum field theory for QED.
- Renormalization of QED using perturbation theory.
- Statistical corrections to the propagating particles.
- Renormalized parameters of QED are treated as effective parameters of the theory.
- Properties of the medium are determined from the particle behavior.
- How neutrinos can couple with the magnetic field (neutral particle).
- Results should be incorporated in experimental/observational data analysis.
- Then application to astrophysics, cosmology and reactor physics.

Introduction

- Interacting fluid media is considered
- Interactions are described as gauge theories
- Renormalization can be proved using
  - Perturbative approach (individual particle approach)
  - Effective potential approach (collective approach)
- Perturbation is described like a series expanded in powers of the coupling constant.
- One loop in Feynman diagrams correspond to a single power of the coupling constant in perturbation theory.

Motivation

- Early Universe (around nucleosynthesis)

  light particles at very high temperature

  ($T \sim 10^{10}$ K or above)

- Astronomical Objects super-massive stars have

  densities $\sim 10^{14} - 10^{16}$ g/cm^3

  $T \sim 10^{10} - 10^{12}$ K

- Magnetic field B is too strong to be ignored in neutron stars and magnetars

  $B \sim 10^{14}$ G

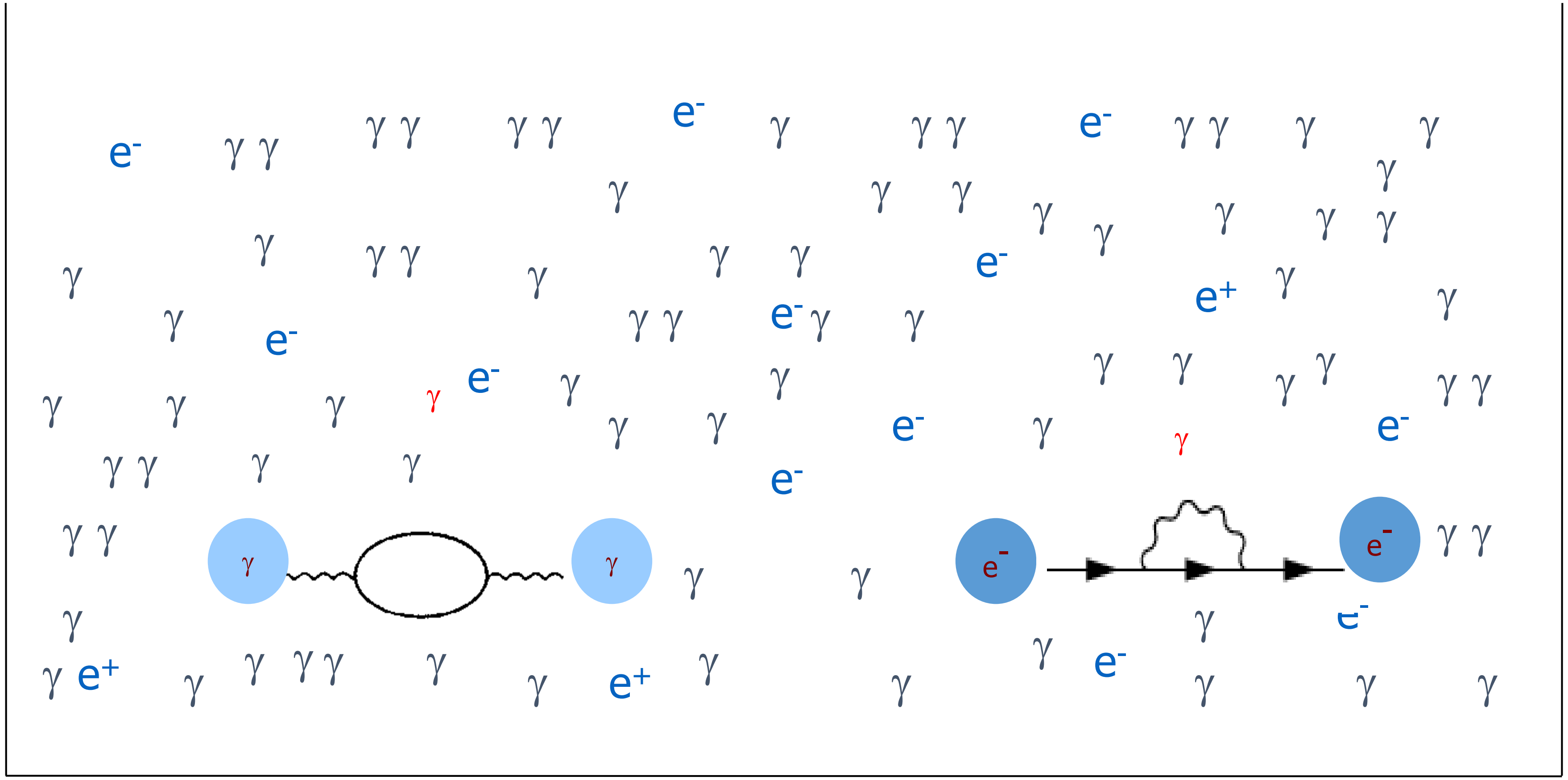
  $\mu_e \sim 200$ MeV (1MeV = 10^{10} K)

Why the medium effects are required

Calculation of QED parameters at extremely high temperature showed that

- QED coupling becomes a function of temperature at the second order of perturbation whereas the first order contributions vanish at low temperature. However the high density effect is non-zero.
- First order perturbation changes the electric permittivity, magnetic permeability and other electromagnetic properties of the medium.
- First order perturbations at extremely high densities and low temperatures indicate the dependence of medium properties on the density or chemical potential of the medium.
- So the electromagnetic properties of a highly dense medium may lead to interesting results, if studied using QED.
- Fluid equations and fluid parameters modification with QED may be interesting .
- Phase transitions between classical fluids/plasmas, relativistic plasmas and superfluids
- Latest discovery of superfluid in the core of neutron star supported the idea that fluid phase depend on the ratio of parameters and not on a single parameter.

Heat Bath



Finite Temperature and Density Effects

- All the physical processes take place in a heat bath comprising all kind of relevant hot particles and antiparticles instead of a vacuum, through the radiative corrections. The probability of exchange of virtual particles with the real particles in the background is incorporated.
- Feynman rules of vacuum theories are used and the basic principles remain unchanged.
- In quantum field theory, the thermal background effects are incorporated via the Fermion and boson propagators only
- Spin statistics of the particles and their properties in vacuum are incorporated as it is.
- The renormalization of electric mass and charge through the self-energy calculation in different frameworks change the physically measurable quantities.

Comparison between formalisms

REAL TIME FORMALISM

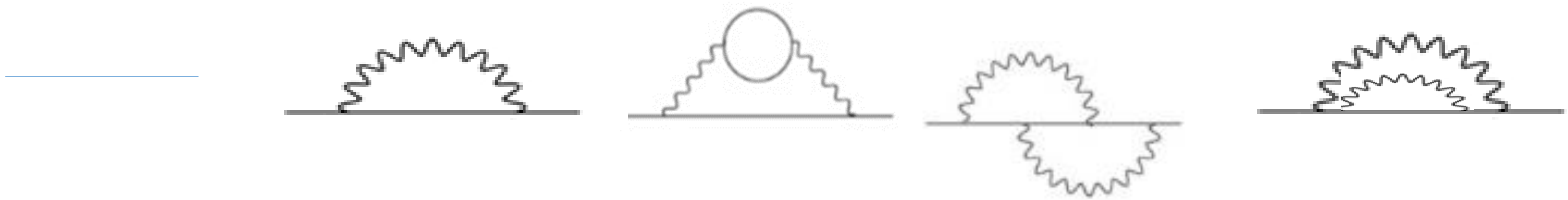
- Gauge invariant exists is induced by hand at the expense of Lorentz invariance.
- Integration on initial and final states hide the path of the system.
- Hard thermal loops appear.
- Segregation is possible between vacuum and statistical contributions, especially at the one loop level.
- Can go to any order of perturbation. But if the perturbative series is convergent and can be truncated wherever, it make sense.
- Renormalizability---Order by order cancellation of singularities can be shown.
- QED works very well with perturbative analysis. So this approach is preferred for the study of electromagnetic properties.

IMAGINARY TIME FORMALISM

- Gauge invariance breaks
- Summation over intermediate states—gives information about the path of the system
- No hard loops
- Statistical effects cannot be separated from the vacuum values.
- All orders of perturbation have to be included.
- Singularities of all orders are cancelled at the same time.
- Only applicable to systems in thermal equilibrium.
- Preferred for non-perturbative analysis of theories where perturbative series does not converge.

Example of Loop diagrams in QED

Self-energy



Vacuum Polarization

$$D_{\mu\nu}(P) = -g_{\mu\nu}\left[\frac{1}{P^2+i\eta} - i\Gamma_b(P)\right],$$

$$S(P) = (P\!\!\!/ + m_0)\left[\frac{1}{P^2-m_0{}^2+i\eta} + i\Gamma_f(P)\right]$$

$$\Gamma_b(P) = 2\pi\delta(P^2)n_b(P),$$

$$\Gamma_f(P) = 2\pi\delta(P^2-m_0{}^2)[\theta(P_0)n_f^+(P) + \theta(-P_0)n_f^-(P)].$$

Fermion propagator

$$S_\beta(p) = (p\!\!\!/ - m)\left\{\frac{i}{p^2-m^2+i\varepsilon} - 2\pi\delta(p^2-m^2)n_F(E_p)\right\},$$

$$n_F(E_p) = \frac{1}{e^{\beta(p.u)}+1},$$

Boson Propagator

$$D_\beta^{\mu\nu}(p) = \left[\frac{i}{k^2+i\varepsilon} - 2\pi\delta(k^2)n_B(k)\right]$$

$$n_B(E_k) = \frac{1}{e^{\beta(k.u)}-1},$$

{u=(1,0,0,0)}

$$\beta = \frac{1}{T}$$

J. F. Donoghue and B. R. Holstein, Phys. Rev. D28 (1983) 340. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.28.340
Edward J. Levinson and David H. Boal, Phys. Rev. D 31, 3280 (1985)
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.31.3280

Renormalization of Gauge Theories

Gauge theories are major interactions theories

Quantum Electrodynamics(QED) is the only abelian gauge (interaction) theory

Renormalization is a procedure to remove singularities of gauge theories

QED coupling is smaller than one in vacuum and the perturbative series converges.

Order by order cancellation of singularities is studied in perturbation theory.

Application of perturbation theory can be used to study QED in statistical medium (described by its composition, temperature, density and magnetic field)

Renormalization Constants of QED

$$\sum(p) = A(p)E\gamma_0 - B(p)\,\boldsymbol{p}\cdot\boldsymbol{\gamma} - C(p)$$

where $A(p)$, $B(p)$, and $C(p)$ are the relevant coefficients and are modified at FTD

$$S^{-1}(p) = (1-A)E\gamma^0 - (1-B)\,p\cdot\gamma - (m-C).$$

$$m_{phys} \cong m + \delta m^{(1)} + \delta m^{(2)}$$

$$\frac{1}{\not{p} - m + i\varepsilon} \longrightarrow \frac{Z_2^{-1}}{\not{p} - m + i\varepsilon}.$$

$$Z_2^{-1} = 1 - A = 1 - \frac{\partial \Sigma(p)}{\partial \not{p}}.$$

Self-energy of electron interactions

- Electrons have a higher physical mass than the electron rest mass due to interactions with its own electric field
- Self-energy/selfenergy 
- Perturbation in vacuum give the additional mass correction due to vacuum fluctuations and thermal energy.
- Renormalized mass = Physical mass of electron

$$m_R \equiv m_{phys} = m + \delta m(T = 0) + \delta m(T).$$

$$m_{phys} \cong m + \delta m^{(1)} + \delta m^{(2)} + \ldots.$$

- New temperature dependent infrared singularities appear due to photon loop
- At high temperatures no additional ultraviolet singularities appear and distribution function provides a cutoff.

First-Order Correction

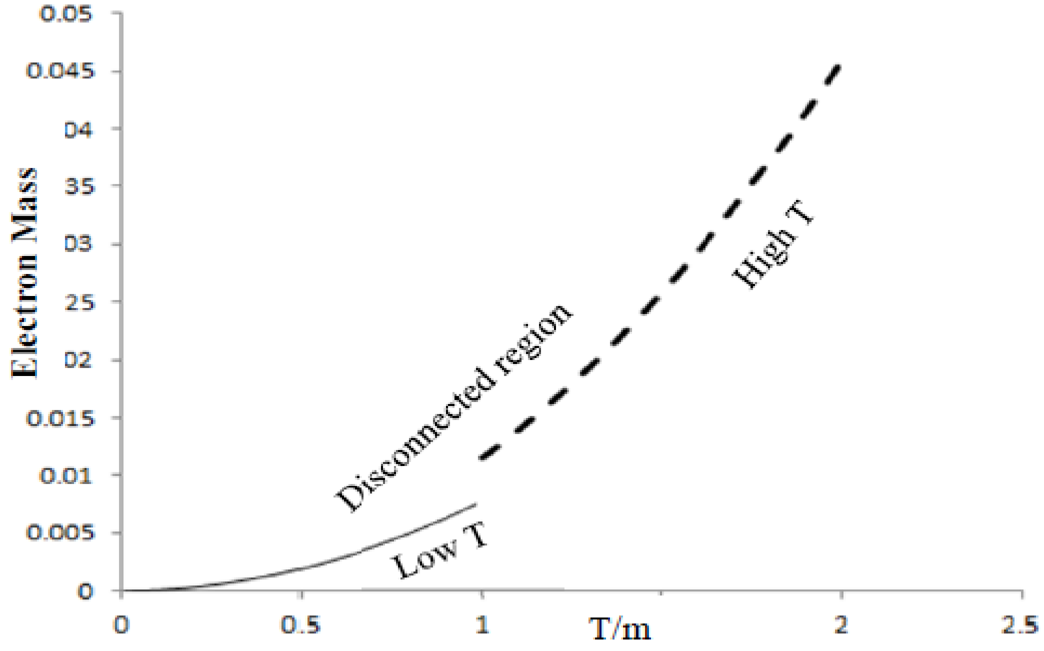
- Low temperature limit $\frac{\delta m}{m} \xrightarrow{T \ll m} \frac{\alpha\pi T^2}{3m^2}.$

- High temperature limit

$$\frac{\delta m}{m} \xrightarrow{T \gg m} \frac{\alpha\pi T^2}{2m^2}.$$

- Difference

$$\Delta\left(\frac{\delta m}{m}\right) = \pm\frac{\alpha\pi T^2}{6m^2} = \pm 3.8\times 10^{-3}\frac{T^2}{m^2}$$



$$\frac{\delta m}{m}(T,\mu) \approx \frac{\alpha\pi T^2}{3m^2}\left[1 - \frac{6}{\pi^2} c(m\beta,\mu)\right] + \frac{2\alpha}{\pi}\frac{T}{m} a(m\beta,\mu) - \frac{3\alpha}{\pi} b(m\beta,\mu).$$

$$Z_2^{-1}(T,\mu) = Z_2^{-1}(T=0,\mu) - \frac{2\alpha}{\pi}\int \frac{dk}{k} n_B(k) - \frac{5\alpha}{\pi} b(m\beta,\mu) + \frac{\alpha}{\pi}\frac{T^2}{E^2}\frac{1}{v} ln \frac{1-v}{1+v} \times \left[c(m\beta,\mu) - \frac{\pi^2}{6} - \frac{m}{T} a(m\beta,\mu)\right]$$

$$Z_3(T,\mu) \approx 1 + \frac{\alpha T^2}{6m^2}\left[\frac{ma(m\beta,\mu)}{T} - c(m\beta,\mu) + \frac{1}{4}\left(m^2 + \frac{\omega^2}{3} b(m\beta,\mu \right)\right]$$

$$Ei(x) = \int_{-x}^{\infty} \frac{e^{-t} dt}{t}$$

$$a(m\beta, \pm\mu) = \ln\left(1 + e^{-(m\pm\mu)}\beta\right)$$

$$b(m\beta, \pm\mu) = \sum_{n=1}^{\infty} (-1)^n e^{\mp n\beta\mu} Ei\left(-nm\beta\right)$$

$$c(m\beta, \pm\mu) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-n(m\pm\mu)\beta}$$

Ahmed, K. and Masood, S.S. (1987) Physical Review D, 35, 1861

Ahmed, K. and Masood, S.S.: Phys. Rev. D 35, (1987) 4020-4023.

In the high chemical-potential limit ($T < m < \mu$), we obtain

$$a(m\beta,-\mu) = \mu - m - \sum_{l=0}^{\infty} \frac{(-1)^l}{\mu^l} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\beta)^{1-l}} e^{-n\beta(m-\mu)}$$

$$b(m\beta,-\mu) = ln\,(\mu/m)$$

$$c(m\beta,-\mu) = \frac{\mu^2 - m^2}{2} - \sum_{l=0}^{\infty} \frac{(-1)^l}{\mu^l} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\beta)^l} e^{-n\beta(m-\mu)}$$

At chemical potential equal to electron mass it gives zero contribution and only thermal corrections can be obtained for high temperatures

Masood, S.S. (1993) Physical Review D, 47, 648. https://journals.aps.org/prd/issues/47/2

Samina Masood and Iram Saleem, IJMPA, Vol. 32, No. 15 (2017)

Electron mass as a function of chemical potential

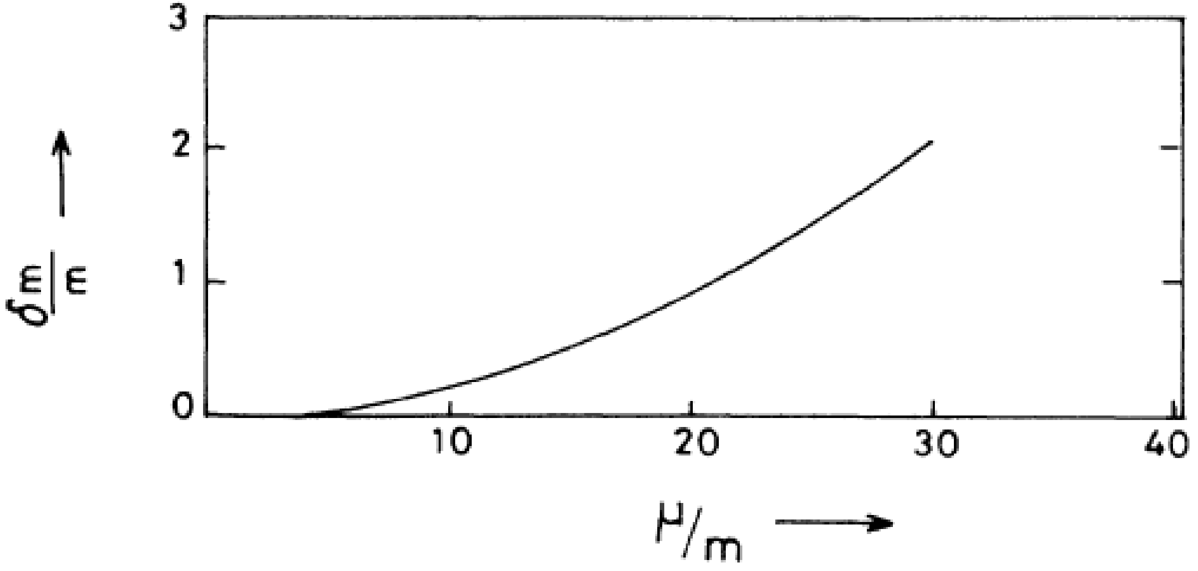


FIG. 1. $\delta m/m$ is plotted against the chemical potential in the limit $T \rightarrow 0$, in the rest frame of electron. It can be clearly seen that $\delta m/m$ increases with the increase in μ/m values.

Samina Saleem (Masood 1987) Physical Review D, 36, 2602. http://dx.doi.org/10.1103/PhysRevD.36.2602

Two-loop corrections to Electron Selfmass

$$\delta m_{\beta}^{(2)} = \frac{1}{m}\left(\Sigma_{T=0}^{(1)} + \Sigma_{T\neq 0}^{(1)}\right)\left(\delta m_{T=0}^{(1)} + \delta m_{T\neq 0}^{(1)}\right)$$

T<<m

$$m_{\text{phys}}^{2} = m_0^2 + \frac{2\alpha}{3}\pi T^2 + \frac{2}{3}\alpha^2 T^2 \left\{ 35 + \frac{3}{v}\ln\frac{1+v}{1-v} + 8\left[\gamma + \ln 2 + \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\ln\left[\frac{nm}{T}\right]\right]\right\},$$

$\gamma = 0.5772$ Euler-Mascheroni constant

Qader, M., Masood, S.S. and Ahmed, K. (1991) Physical Review D, 44, 3322.
Qader, M., Masood, S.S. and Ahmed, K. (1992) Physical Review D, 46, 5633

Two-loop corrections to Electron Selfmass

Initially , it was proposed that at high temperature we may write thermal effect in the series expansion

$$\Sigma_\beta(p) \sim T \sum_{n=1} c_n \alpha^n \left[\frac{T}{p} \ln \frac{E+p}{E-p} \right]^n$$

But it was later found to be much more complicated

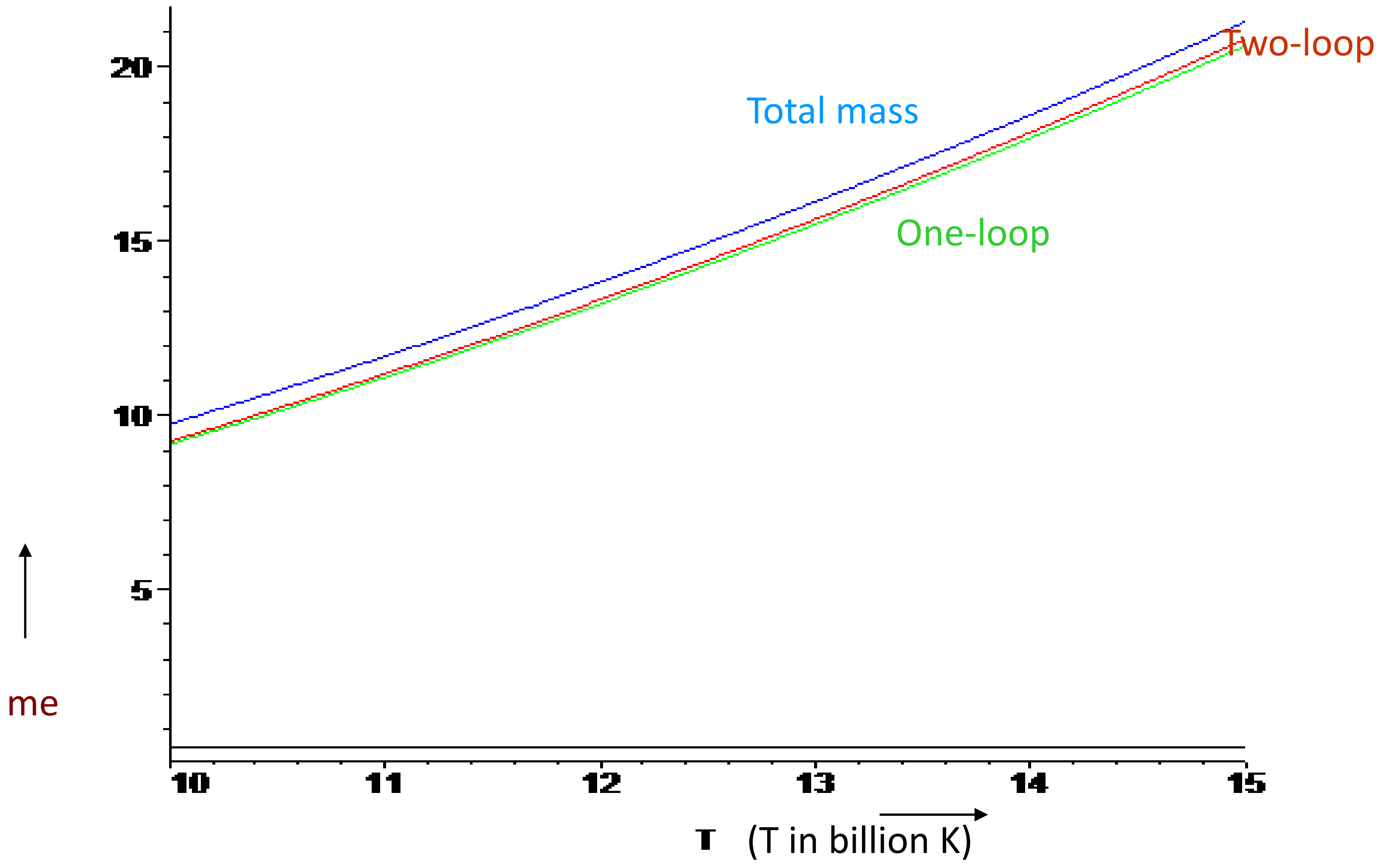
Duane A. Dicus, David Down, and Edward W. Kolb, Nucl. Phys. B223, 525 (1983).
Qader, M., Masood, S.S. and Ahmed, K. (1991) Physical Review D, 44, 3322. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.44.3322
Qader, M., Masood, S.S. and Ahmed, K. (1992) Physical Review D, 46, 5633. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.46.5633

$$\frac{\delta m^{(2)}}{m} \underset{T \gg m_e}{\sim} \alpha^2 \left(\frac{T}{m}\right)^3 \left[\frac{4}{\pi^2}\left\{\zeta(3)\frac{E}{m}\left[\left\{20\left(1+\frac{1}{v}\right)+7v\right\}\ln\frac{1+v}{1-v}-\frac{4E}{m}\right]\right.\right.$$

$$\left.+2\left[\frac{10m}{Ev}-\frac{E}{m}+\frac{7E}{mv}(1+v^2)\right]F_1+\frac{\pi m}{6Ev}F_2\right],$$

Qader, M., Masood, S.S. and Ahmed, K. (1992) Physical Review D, 46, 5633.
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.46.5633

Comparison of one and two-loop corrections at High T



Wavefunction Renormalization Constant

$$Z_2^{-1}(m\beta) = Z_2^{-1}(T=0) - \frac{2\alpha}{\pi}\int_0^\infty \frac{dk}{k} n_B(k) - \frac{3\alpha}{\pi} b(m\beta)$$

$$+ \frac{\alpha T^2}{\pi v E^2} \ln \frac{1+v}{1-v} \left\{ \frac{\pi^2}{6} + m\beta a(m\beta) - c(m\beta) \right\},$$

Low T

$$Z_2^{-1} = Z_2^{-1}(T=0) - \frac{2\alpha}{\pi}\int \frac{dk}{k} n_B(k) - \frac{\alpha\pi T^2}{3E^2},$$

High T

$$Z_2^{-1} = Z_2^{-1}(T=0) - \frac{2\alpha}{\pi}\int \frac{dk}{k} n_B(k) - \frac{\alpha\pi T^2}{2E^2},$$

K. Ahmed and Samina Saleem (Masood), Phys. Rev. D 35, 1861 (1987).

$$Z_2^{-1} \xrightarrow{T \gg m} 1 - \alpha\left[\frac{2I_A}{\pi} + \frac{1}{4\pi}\left(\frac{3}{\varepsilon} - 4\right) + \frac{4\pi T^2}{3}\right] - \alpha^2\left[\frac{1}{4\pi^3}\left\{\frac{3}{\varepsilon}(I_A + J_A) + (3I_A + 5J_A) - \frac{8T^2}{3m^2}\right\}\right.$$

$$+ \frac{1}{8}\sum_{n,r,s=1}^{\infty}(-1)^r T\left\{e^{-r\beta E}\left[f_+(s,r)\frac{\gamma\cdot\mathbf{p}}{E^2v^2} - \frac{I_B I_C}{64\pi^2} + h(p,\gamma)\left\{f_-(n,r)\frac{I_C}{8\pi} - f_-(s,r)\right\} + f_+(n,r)\frac{\gamma\cdot\mathbf{p}}{E^2v^2}\frac{I_B}{8\pi}\right.\right.$$

$$\left.\left.\left. + \left\{\frac{2}{m}h(p,\gamma) - \frac{\gamma\cdot\mathbf{p}}{E^2v^2}\right\}f_-(s,r)\right]\right\}\right].$$

Mahnaz Haseeb and Samina Masood, Phys Lett. B704 (2011) 66-73

Effect of Electron mass on different parameters

- Magnetic moment of electron
- Electromagnetic properties of particles that propagate in the presence of electron
- Effect on weak interaction processes (involving electrons) such as beta decay
- Decay rates and crossections of particles

Electromagnetic vertex

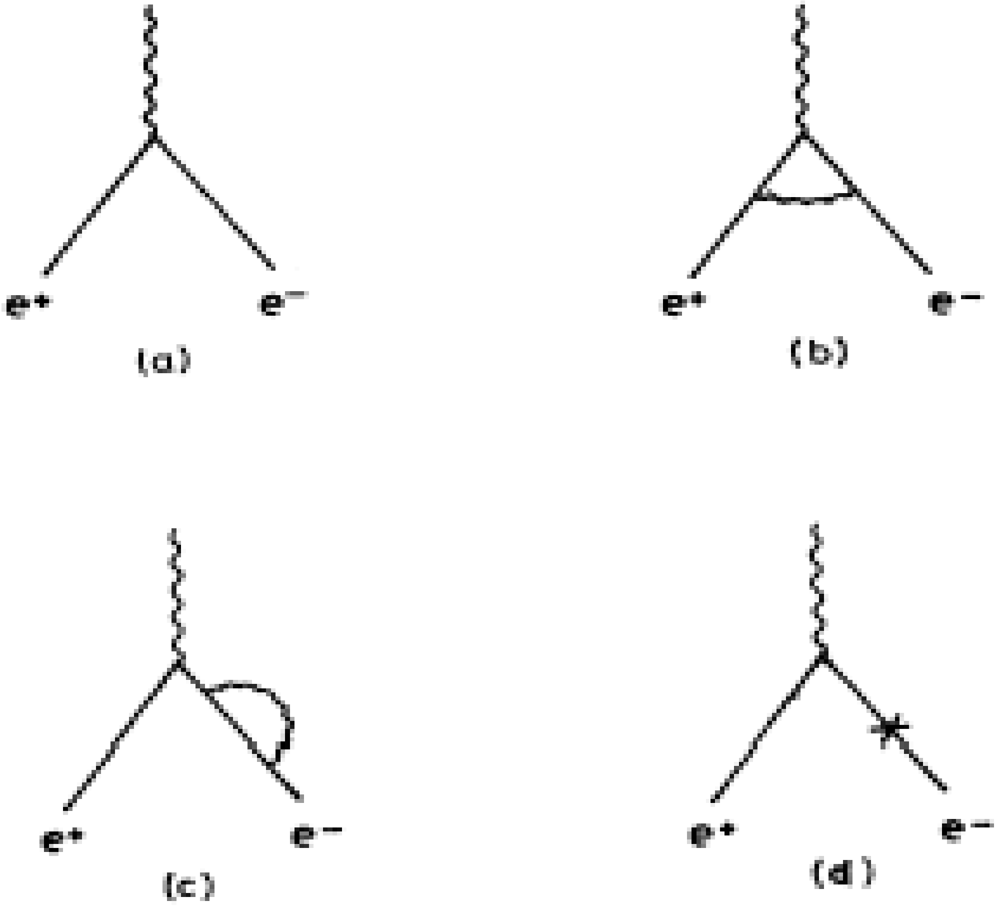


FIG. 3. The first-order radiative corrections to the electromagnetic vertex. (a) is the bare vertex, (b) the vertex corrections, (c) the self-energy correction to one of the legs, and (d) the mass counterterm insertion on one of the legs.

Z-Decay

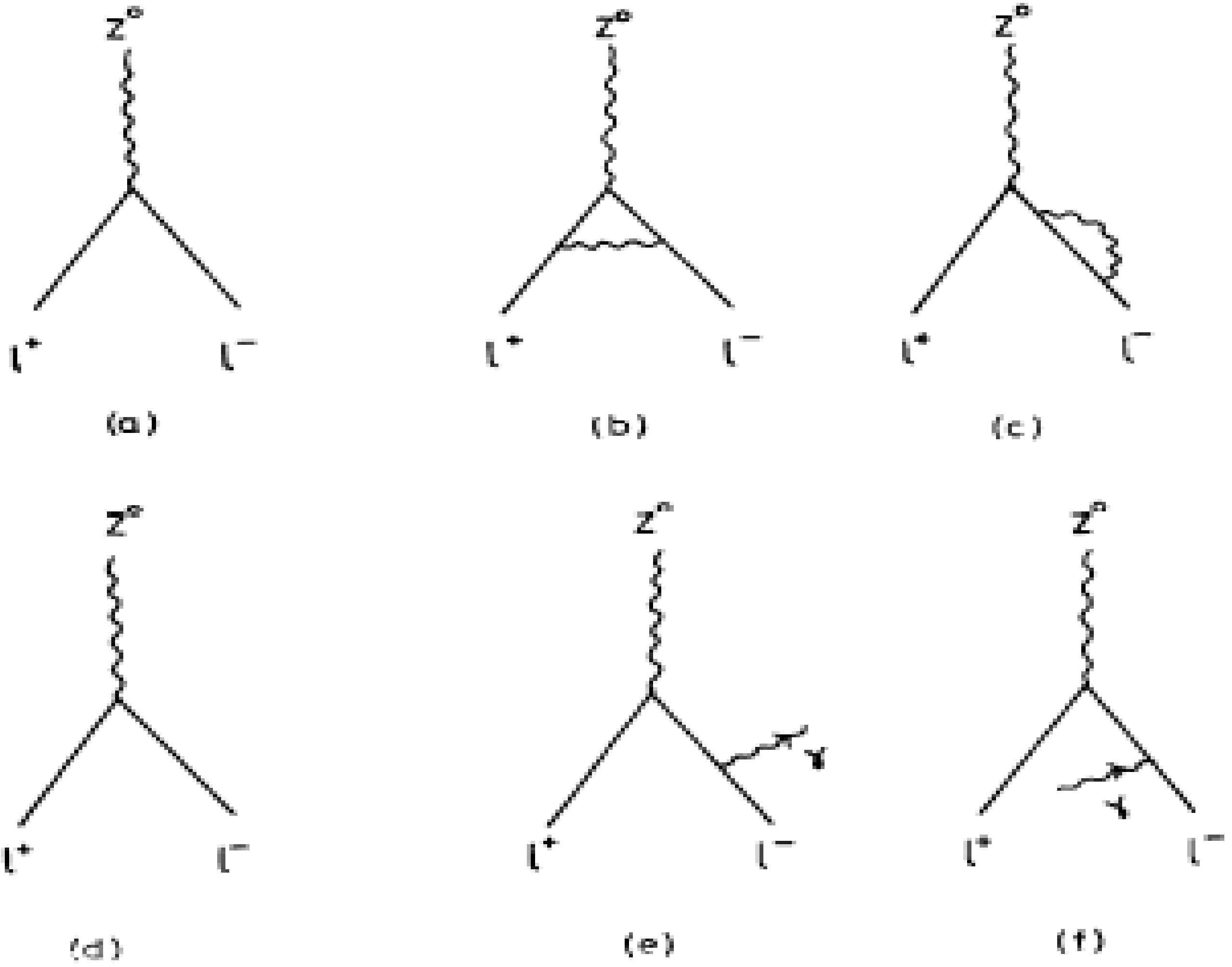


FIG. 1. The diagrams for the decay $Z^0 \to l^+ l^-$ up to the first order in α. (a) is the tree-level diagram, (b) is the vertex correction, (c) is the self-energy correction, (d) is the mass counterterm, (e) shows the bremsstrahlung emission and (f) the bremsstrahlung absorption.

Phase space contribution

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}[1-n_F(p\pm\mu)]$$

for fermions and

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k}[1+n_B(k)]$$

$$\Delta\Gamma_{\rm PS} = \frac{4\alpha}{\pi v^2}\left[\frac{9m_l^2}{m_Z^2}\tilde{b}(m_l\beta,\mu) - \frac{6m_l T}{m_Z^2}\tilde{a}(m_l\beta,\mu) - \frac{T^2}{m_Z^2}\left[1-\frac{6}{\pi^2}\tilde{c}(m_l\beta,\mu)\right]\right]. \qquad (2.13)$$

Higgs Decay --- example

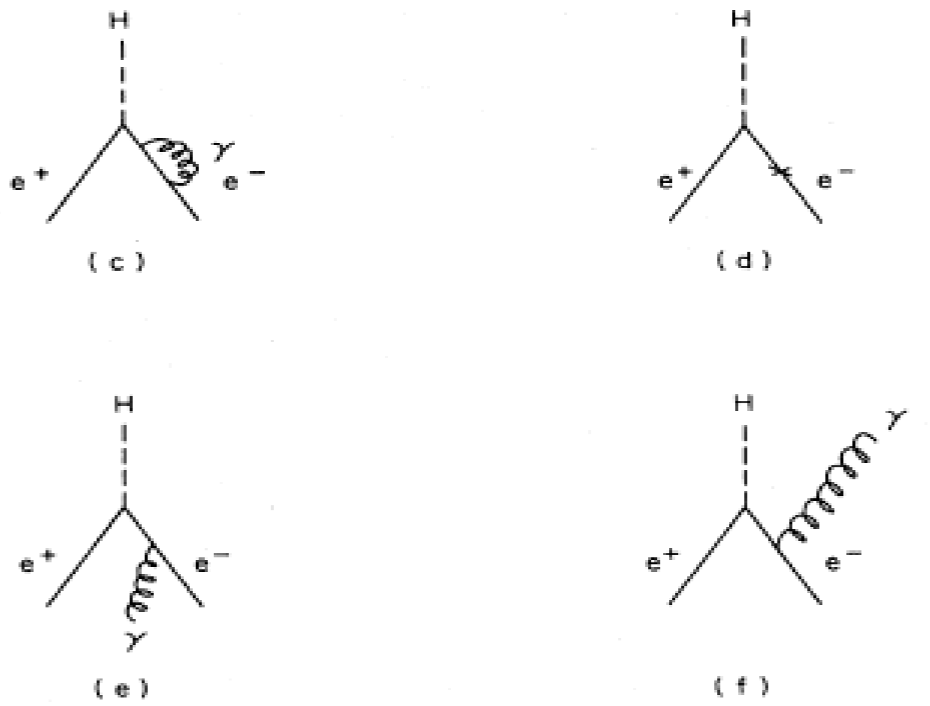


FIG. 2. Diagrams for $H \to e^+e^-$ to order α. Diagram (a) gives the bare coupling, (b) is the vertex correction, (c) and (d) are the self-energy and mass-counterterm contributions, respectively, while (e) and (f) correspond to real emission and absorption.

Taking the modulus squared and summing over spins

$$\sum_s |M|^2 = 4e^2 g^2 \left[(p_1 \cdot p_2 - m^2) \left[\frac{2 p_1 \cdot p_2}{p_1 \cdot k \, p_2 \cdot k} - \frac{m^2}{(p_1 \cdot k)^2} - \frac{m^2}{(p_2 \cdot k)^2} \right] + \left[2 p_1 \cdot p_2 - m^2 \frac{p_1 \cdot k}{p_2 \cdot k} - m^2 \frac{p_2 \cdot k}{p_1 \cdot k} \right] \left[\frac{1}{p_1 \cdot k} + \frac{1}{p_2 \cdot k} \right] + \left[\frac{p_1 \cdot k}{p_2 \cdot k} + \frac{p_2 \cdot k}{p_1 \cdot k} + 2 \right] \right] .$$

$$\Delta\Gamma_{\text{tot}} \simeq 2\Gamma_0 \left\{ \frac{\alpha T}{\pi E^2 v^2} \left[-m + \frac{1}{v} \left[m - \frac{1+v^2}{2} E \right] \ln \frac{1-v}{1+v} \right] a(m\beta) + \frac{\alpha}{\pi v^2} \left[3v^2 - v \ln \frac{1-v}{1+v} + \frac{11}{2} \frac{m^2}{E^2} - \frac{1}{2} \right] b(m\beta) - \frac{\alpha T^2}{\pi E^2 v^3} \left[\ln \frac{1-v}{1+v} + 3v \right] c(m\beta) \right\} .$$

In order to calculate the contribution of stimulated emission or absorption from the heat bath, one must integrate (2.12) over the three-body phase space and carry out the Dalitz-plot analysis as done by DH. Since for bremsstrahlung, the relevant temperature correction accounts for the corresponding correction to the density of final states of photons: namely,

$$\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[1+n_B(E_p)]\ ; \tag{2.13}$$

therefore, no modification is expected in this case in the DH results. We shall return to this question in the next subsection dealing with the decay rate of $H\to e^+e^-$.

$$\Delta\Gamma_{\rm PS}\simeq 36\frac{\alpha}{\pi v^2}\frac{m^2}{m_H{}^2}b(m\beta)-24\frac{\alpha}{\pi v^2}\frac{mT}{m_H{}^2}a(m\beta)$$

$$-\frac{4\alpha\pi}{v^2}\frac{T^2}{m_H{}^2}\left[1-\frac{6}{\pi^2}c(m\beta)\right]+O(T^4)\ .$$

$$\Delta\Gamma_{\text{tot}} \simeq \frac{4\alpha\Gamma_0}{\pi}\left\{\frac{T}{m_Z v}\left[\left(1-\frac{2m_l}{m_Z}+\frac{4m_l^2}{m_Z^2 v^2}\right)\ln\frac{1+v}{1-v}-\frac{2m_l}{m_Z v}\left(3+\frac{4m_l}{m_Z^2 v^3}\right)\right]\right.$$

$$\times\tilde{a}(m_l\beta,\mu)+\left[1+\frac{2m_l^2}{m_Z^2 v^2}-\frac{1}{2v}\left(1-\frac{5}{4}\frac{m_l^2}{m_Z^2 v^2}+\frac{m_l^2}{4m_Z^2 v^4}\right)\ln\frac{1+v}{1-v}\right]$$

$$\left.\times\tilde{b}(m_l\beta,\mu)+\frac{2T^2}{m_Z^2 v}\left[\left(1+\frac{m_l^2}{m_Z^2 v^3}\right)\ln\frac{1+v}{1-v}+\frac{1}{v}\left(3-\frac{2m_l^2}{m_Z^2 v^2}\right)\right]\tilde{c}(m_l\beta,\mu)\right\}$$

$$\tilde{a}(m_l\beta,\mu)=\frac{1}{2}\ln\left[\left(1+e^{-(m_l-\mu)\beta}\right)\left(1+e^{-(m_l+\mu)\beta}\right)\right]$$

$$\tilde{b}(m_l\beta,\mu)=\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\mathrm{Ei}(-nm_l\beta)\ ,$$

$$\tilde{c}(m_l\beta,\mu)=\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\frac{e^{-nm_l\beta}}{n^2}\ .$$

$$\Delta\Gamma_{\text{tot}}\xrightarrow{T>m_e}-\frac{2\pi\alpha\Gamma_0}{3}\frac{T^2}{m_Z^2 v^3}\left[\left(1+\frac{m_l^2}{m_Z^2 v^3}\right)\ln\frac{1+v}{1-v}+\frac{1}{v}\left(3-\frac{2m_l^2}{m_Z^2 v^2}\right)\right]$$

Mahnaz Haseeb and Samina Masood, Phys Rev. D46 (1992) 5110

TABLE II. Temperature corrections to the decay rate of $H \rightarrow e^+e^-$ are computed for different values of temperature at $m_H = 1$ GeV/c^2. The table shows that these corrections become significant for certain temperatures.

$m\beta$	T (MeV and K)	$\Delta\Gamma_{tot}/\Gamma_0$	Percentage change in the decay rate
0.001	$500/5.93\times10^{12}$	0.2629	26.29
0.005	$100/1.19\times10^{12}$	0.2275	22.75
0.01	$50/5.93\times10^{11}$	0.1834	18.34
0.05	$10/1.19\times10^{11}$	0.1147	11.47
0.10	$5/5.93\times10^{10}$	0.0843	8.43
0.5	$1/1.19\times10^{10}$	0.0326	3.26
1.0	$0.5/5.93\times10^{9}$	0.0147	1.47
2.0	$0.25/2.96\times10^{9}$	0.0037	0.37

Lamb Shift

The lamb shift is a purely measurable perturbative effect, which causes a small change in wavelength. Knight has looked at the thermal effects on the Lamb shift. A small shift in the bound state energies is calculated from the regular quantum mechanical energy values corresponding to the integral values on n such that:

$$E_n = -\frac{me^4}{8(nh\varepsilon_0)^{\,2}}$$

$$E = E_f - E_i = \left(\frac{me^4}{8(nh\varepsilon_0)^2}\right)\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$$

The change in the electron mass will affect the energies of the energy levels. Comparing these formulae with the Rydberg formula gives a shift on energy between successive energy levels as

$$R_H = -\frac{me^4}{8h^3{\varepsilon_0}^2}$$

On the other hand the total energy shift due to the vacuum polarization is determined as:

$$\Delta E = \int d^3r \;\Delta V(r)|\psi(r)|^2$$

1. P.L.Knight, J. Phys. A: Gen. Phys., Vol. *5,* March (1972.),`Effects of External Fields on the Lamb Shift' (417-425)

$$\Delta V(r) = \frac{e^2}{(2\pi)^3} \int d^3q \, e^{iq\cdot r} \left[\frac{\widetilde{\Pi}(q)}{q^2} \right]$$

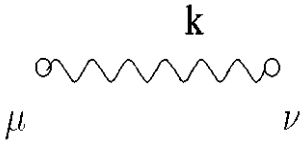
$$\Delta E_n = \frac{-4\alpha^5 m}{15\pi n^3}$$

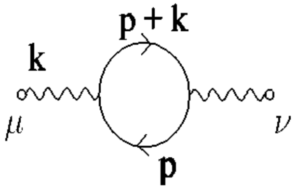
n is the Landau number and is associated with the Landau levels of energy for the Fermi gas of electron. This correction may not be relevant for atoms for temperature dependent electron mass as the existence of atoms is not expected at 10^{10} K. However, high chemical potential is not necessarily ignorable for large molecules.

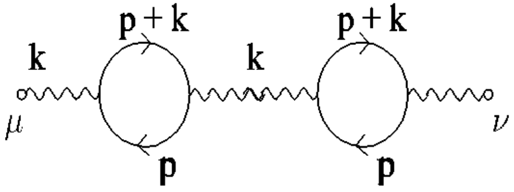
Lecture 2

- Light is the major source of information for astronomical observations.
- Properties of light can be related to the properties of their source.
- Information of the source can help to understand the origin of the radiation
- It may provide a little more information about the structure and composition of the source of radiation.
- The nature of the wave can help to determine the collective properties of the medium as well.

Renormalization of Photon Propagator

$D'_{\mu\nu}(k) =$ 

$+$ 

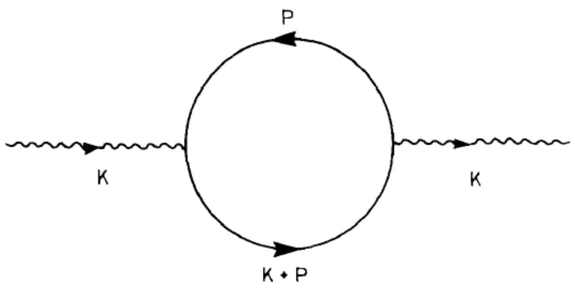
$+$ 

$+ \ \ldots$

$$= \frac{1}{D^{-1}{}_{\mu\nu} - i\Pi_{\mu\nu}} \; = \; \frac{-i}{k^2 - \mathcal{O}(\Lambda^2)}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

Calculation of Charge Renormalization

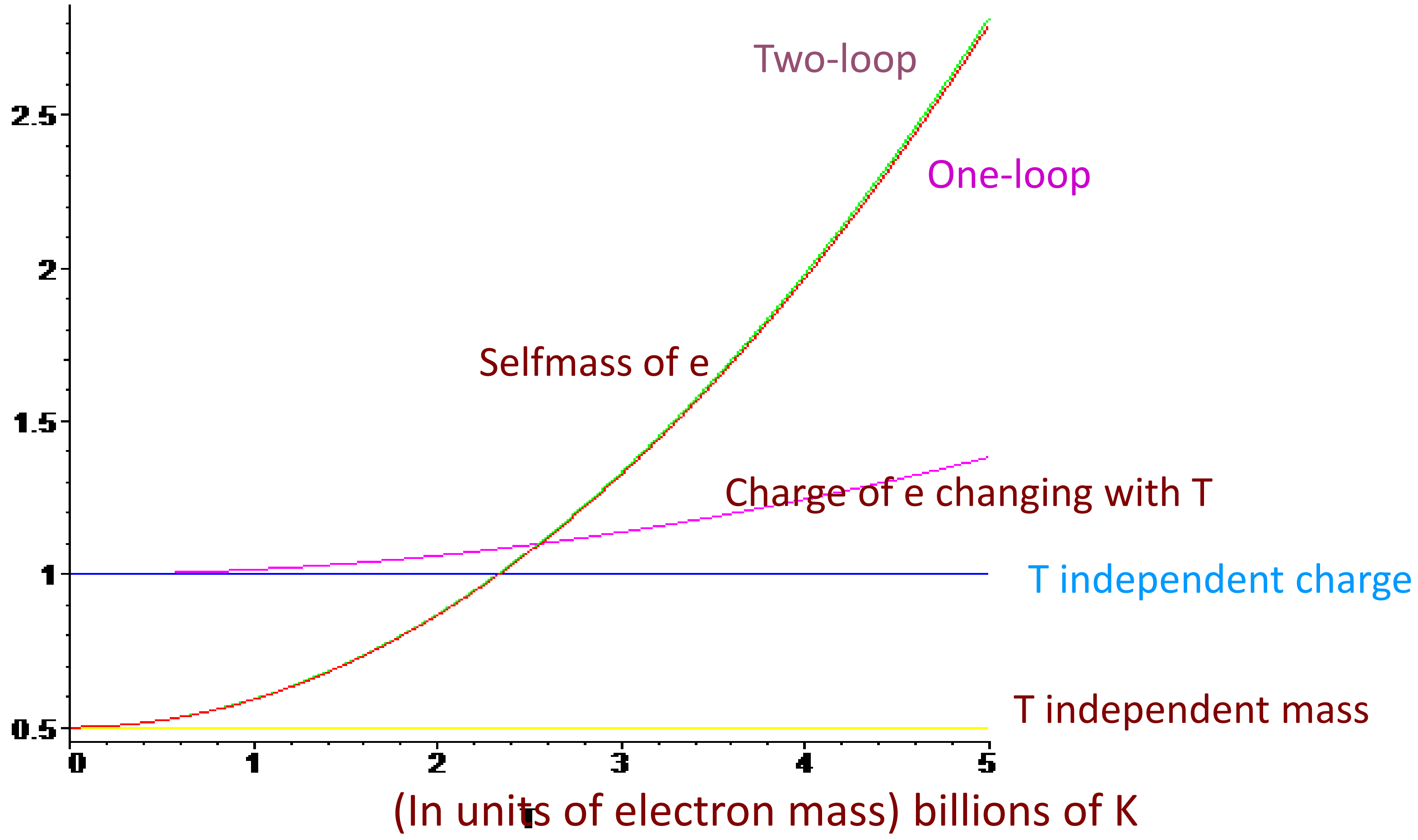
$$\pi_{\mu\nu}(K,\mu) = ie^2 \int \frac{d^4p}{(4\pi)^4} \mathrm{Tr}\{\gamma_\mu(\not{p}+\not{K}+m)\gamma_\nu(\not{p}+m)\}\left[\frac{1}{(p+K)^2-m^2} + \Gamma_F(p+K,\mu)\right]\left[\frac{1}{p^2-m^2}+\Gamma_F(p,\mu)\right],$$



$$\Gamma_F(p,\mu) = 2\pi i\delta(p^2-m^2)[\theta(p_0)n_F(p,\mu)+\theta(-p_0)n_F(p,-\mu)]$$

$$\pi^{\beta}_{\mu\nu}(K,\mu) = -\frac{2\pi e^2}{2}\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\{\gamma_\mu(\not{p}+\not{K}+m)\gamma_\nu(\not{p}+m)\}$$
$$\times\left[\frac{\delta[(p+K)^2-m^2]}{p^2-m^2}\{n_F(p+K,\mu)+n_F(p+K,-\mu)\} + \frac{\delta(p^2-m^2)}{(p+K)^2-m^2}\{n_F(p,\mu)+n_F(p,-\mu)\}\right].$$

Thermal Corrections



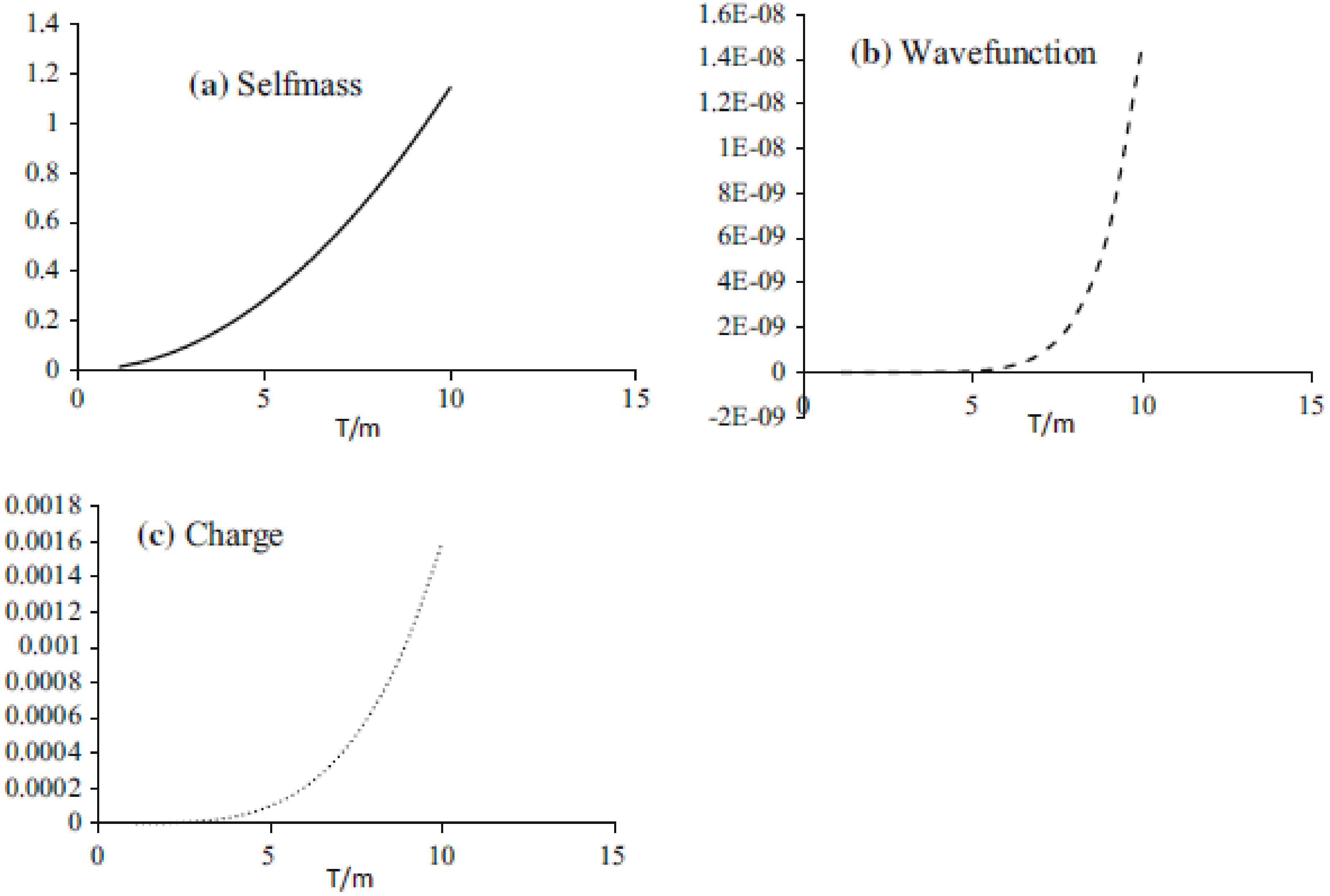


Fig. 2 The properties of the electrons as a function of temperature (in units of m). The thermal corrections to a electron mass; b wave function; and c charge

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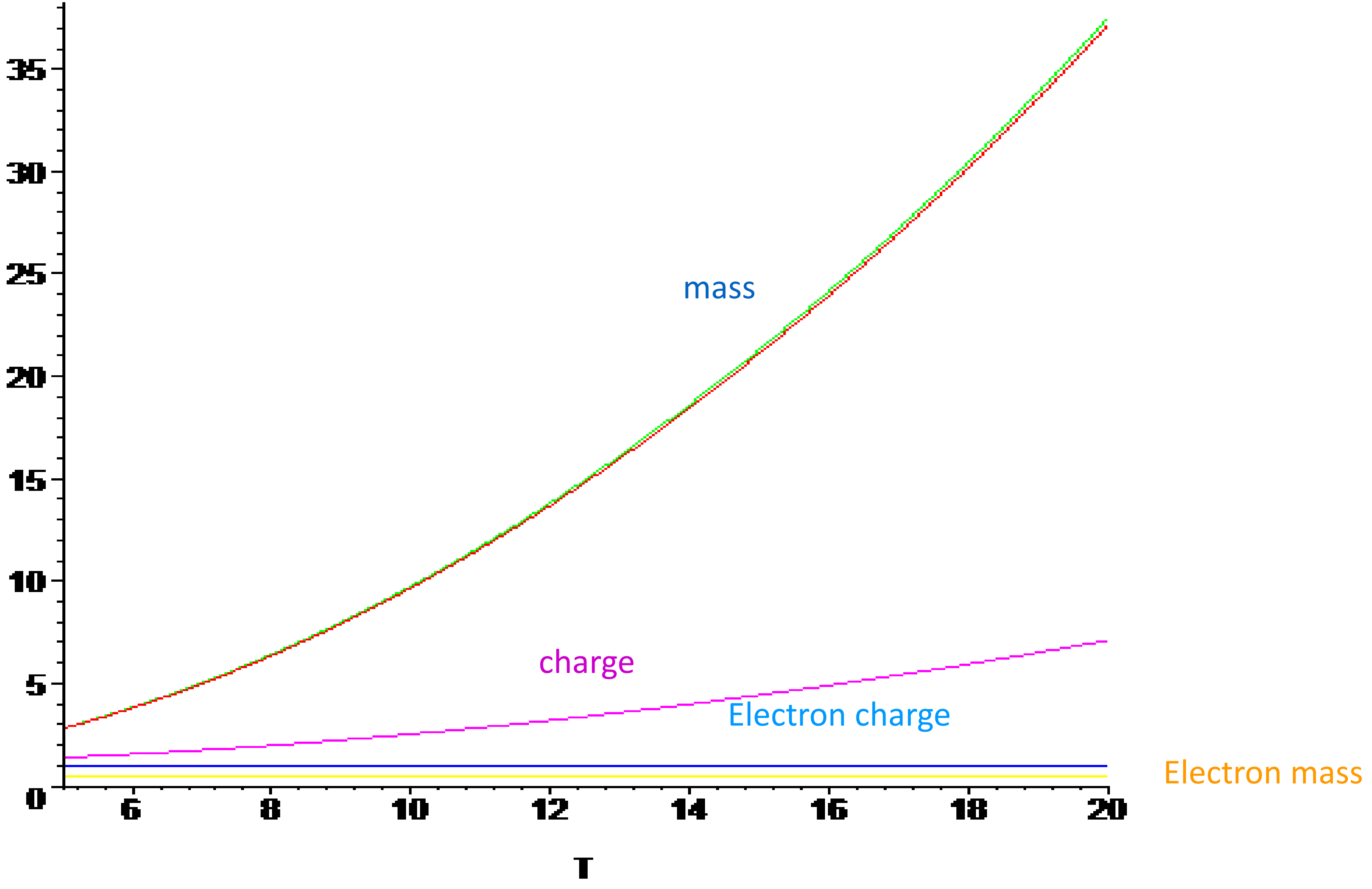
ELECTRON MASS and CHARGE at High T



$$\pi_L(k,\omega) = -\frac{K^2}{k^2} u^\mu u^\nu \pi_{\mu\nu}$$

$$\pi_T(k,\omega) = -\tfrac{1}{2}\pi_L + \tfrac{1}{2} g^{\mu\nu}\pi_{\mu\nu}.$$

$$c(m\beta) \xrightarrow{m\beta \ll 1} -\frac{\pi^2}{12},$$

$$\pi_L \simeq \frac{e^2T^2}{3}\left(1-\frac{\omega^2}{k^2}\right)\left[1-\frac{\omega}{2k}\ln\frac{\omega+k}{w-k}\right]$$

$$\pi_T \simeq \frac{e^2T^2}{6}\left[\frac{\omega^2}{k^2}+\left(1-\frac{\omega^2}{k^2}\right)(\omega/2k)\ln\frac{\omega+k}{\omega-k}\right],$$

.K.Ahmed and Samina Saleem, Ann.Phys.164, (1991) 460.

$$\pi_{\mu\nu}(K) = \pi_{\mathrm{T}}(k, \omega)\, P_{\mu\nu} + \pi_{\mathrm{L}}(k, \omega)\, Q_{\mu\nu}.$$

$$P_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{K}_{\mu}\tilde{K}_{\nu}/k^2$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}$$

$$\tilde{K}_{\mu} = K_{\mu} - \omega u_{\mu}$$

$$Q_{\mu\nu} = -\frac{1}{k^2K^2}(k^2u_{\mu} + \omega\tilde{K}_{\mu})(k^2u_{\nu} + \omega\tilde{K}_{\nu}),$$

$$P^{\mu}_{\nu}P^{\nu}_{\alpha} = P^{\mu}_{\alpha},\; Q^{\mu}_{\alpha} = Q^{\mu}_{\nu}Q^{\nu}_{\alpha},$$

$$K_{\mu}P^{\mu}_{\nu} = 0, \qquad K_{\mu}Q^{\mu}_{\nu} = 0.$$

$$
\pi_{\mu\nu} = \begin{pmatrix}
-\frac{k^2}{K^2}\pi_L & -\frac{i\omega k_1}{K^2}\pi_L & -\frac{i\omega k_2}{K^2}\pi_L & -\frac{i\omega k_3}{K^2}\pi_L \\
-\frac{i\omega k_1}{K^2}\pi_L & \left(-1-\frac{k_1^2}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_1^2}{k^2K^2}\right)\pi_L & \left(-\frac{k_1k_2}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_1k_2}{k^2K^2}\right)\pi_L & \left(-\frac{k_1k_3}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_1k_3}{k^2K^2}\right)\pi_L \\
-\frac{i\omega k_2}{K^2}\pi_L & \left(-\frac{k_1k_2}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_1k_2}{k^2K^2}\right)\pi_L & \left(-1-\frac{k_2^2}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_2^2}{k^2K^2}\right)\pi_L & \left(-\frac{k_2k_3}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_2k_3}{k^2K^2}\right)\pi_L \\
-\frac{i\omega k_3}{K^2}\pi_L & \left(-\frac{k_1k_3}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_1k_3}{k^2K^2}\right)\pi_L & \left(-\frac{k_2k_3}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_2k_3}{k^2K^2}\right)\pi_L & \left(-1-\frac{k_3^2}{k^2}\right)\pi_T + \left(\frac{\omega^2 k_3^2}{k^2K^2}\right)\pi_L
\end{pmatrix}.
$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 - \dfrac{k_1^2}{k^2} & -\dfrac{k_1 k_2}{k^2} & -\dfrac{k_1 k_3}{k^2} \\ 0 & -\dfrac{k_1 k_2}{k^2} & -1 - \dfrac{k_2^2}{k^2} & -\dfrac{k_2 k_3}{k^2} \\ 0 & -\dfrac{k_1 k_3}{k^2} & -\dfrac{k_2 k_3}{k^2} & -1 - \dfrac{k_3^2}{k^2} \end{pmatrix}$$

$$Q_{\mu\nu} = \begin{pmatrix} -\dfrac{k^2}{K^2} & -\dfrac{i\omega k_1}{K^2} & -\dfrac{i\omega k_2}{K^2} & -\dfrac{i\omega k_3}{K^2} \\ -\dfrac{i\omega k_1}{K^2} & \dfrac{\omega^2 k_1^2}{k^2 K^2} & \dfrac{\omega^2 k_1 k_2}{k^2 K^2} & \dfrac{\omega^2 k_1 k_3}{k^2 K^2} \\ -\dfrac{i\omega k_2}{K^2} & \dfrac{\omega^2 k_1 k_2}{k^2 K^2} & \dfrac{\omega^2 k_2^2}{k^2 K^2} & \dfrac{\omega^2 k_2 k_3}{k^2 K^2} \\ -\dfrac{i\omega k_3}{K^2} & \dfrac{\omega^2 k_1 k_3}{k^2 K^2} & \dfrac{\omega^2 k_2 k_3}{k^2 K^2} & \dfrac{\omega^2 k_3^2}{k^2 K^2} \end{pmatrix},$$

$$\pi_{\mathrm{L}} \simeq \frac{4e^2}{\pi^2}\left(1-\frac{\omega^2}{k^2}\right)\left[\left(1-\frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k}\right)\left(\frac{m\tilde{a}(m\beta,\mu)}{\beta}-\frac{\tilde{c}(m\beta,\mu)}{\beta^2}\right)\right.$$
$$\left.+\frac{1}{4}\left(2m^2-\omega^2+\frac{11k^2+37\omega^2}{72}\right)\tilde{b}(m\beta,\mu)\right]$$

$$\pi_{\mathrm{T}} \simeq \frac{2e^2}{\pi^2}\left[\left\{\frac{\omega^2}{k^2}+\left(1-\frac{\omega^2}{k^2}\right)\ln\frac{\omega+k}{\omega-k}\right\}\left(\frac{m\tilde{a}(m\beta,\mu)}{\beta}-\frac{\tilde{c}(m\beta,\mu)}{\beta^2}\right)\right.$$
$$\left.+\frac{1}{8}\left\{2m^2+\omega^2+\frac{107\omega^2-131k^2}{72}\right\}\tilde{b}(m\beta,\mu)\right],$$

$$\tilde{a}(m\beta,\mu)=\tfrac{1}{2}\ln\{(1+e^{-\beta(m+\mu)})(1+e^{-\beta(m-\mu)})\}$$

$$\tilde{b}(m\beta,\mu)=\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\,Ei(-nm\beta) \qquad Ei(-x)=-\int_x^{\infty}\frac{e^{-t}}{t}\,dt.$$

$$\tilde{c}(m\beta,\mu)=\sum_{n=1}^{\infty}(-1)^n\frac{e^{-nm\beta}}{n^2}\cosh(n\beta\mu)$$

Charge Renormalization Constant

$$Z_3 \cong 1 + \frac{2e^2}{\pi^2}\left[\frac{ma(m\beta,\mu)}{\beta} - \frac{c(m\beta,\mu)}{\beta^2} + \frac{1}{4}\left(m^2 + \frac{1}{3}\omega^2\right) b(m\beta,\mu)\right].$$

Dynamically Generated Mass of Photon

$$K_\mu(\omega, \mathbf{k}) \to 0. \qquad K_0 = \omega = 0 \text{ i}$$

$$\lim_{\mathbf{k}\to 0} \pi_L(0, \mathbf{k}) \equiv K_L^2 = \frac{4e^2}{\pi^2}\left[\frac{m\tilde{a}(m\beta, \mu)}{\beta} - \frac{\tilde{c}(m\beta, \mu)}{\beta^2} + \frac{m^2}{2} \cdot \tilde{b}(m\beta, \mu)\right]$$

$$\lim_{\mathbf{k}\to 0} \pi_T(0, \mathbf{k}) \equiv K_T^2 = 0.$$

Plasma Screening Mass

$$\lambda_{\rm D} = \Pi_{\rm L}.$$

$$\lambda_{\rm D} = \left(\frac{\varepsilon k_{\rm B} T}{\sum_{j=1}^{N} n_j^0 q_j^2} \right)^{1/2}.$$

$$\nu_{\rm prop} = \sqrt{\frac{1}{\mu(K)\epsilon(K)}},$$

$$\omega_D = \frac{2\pi\,\nu_{\rm prop}}{\lambda_D} = 2\pi\,K_L\,\nu_{\rm prop},$$

$$r_i = \frac{c}{\nu_{\rm prop}} = \sqrt{\frac{\mu(K)\epsilon(K)}{\mu_0(K)\epsilon_0(K)}}.$$

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.

if we set $\omega = |\mathbf{k}| = k$ with $k \to 0$ then

$$\lim_{k \to 0} \pi_T(k, \mathbf{k}) \equiv \omega_T^2 = \frac{2e^2}{\pi^2}\left[\frac{m\tilde{a}(m\beta, \mu)}{\beta} - \frac{\tilde{c}(m\beta, \mu)}{\beta^2} + \frac{m^2}{4}\tilde{b}(m\beta, \mu)\right]$$

$$\lim_{\mathbf{k} \to 0} \pi_L(|\mathbf{k}|, \mathbf{k}) \equiv \omega_L^2 = 0.$$

$$\omega_T^2 = \frac{2e^2}{\pi^2}\left[\frac{m\tilde{a}}{\beta} - \frac{\tilde{c}}{\beta^2} + \frac{m^2}{4}\tilde{b}\right]$$

At very high T

$$D'_{F\mu\nu} = \frac{P_{\mu\nu}}{K^2 - \pi_{\mathrm{T}}} + \frac{Q_{\mu\nu}}{K^2 - \pi_{\mathrm{L}}}.$$

$$\omega_{\mathrm{T}}^2 \xrightarrow[m\beta \to 0]{} \frac{2e^2}{\pi^2}(-1/\beta^2) \lim_{m\beta \to 0} c(m\beta) = \frac{2e^2}{\pi^2}(-1/\beta^2)\left(\frac{-\pi^2}{12}\right) = \frac{e^2 T^2}{6}.$$

$$K_{\mathrm{L}}^2 \xrightarrow[\substack{m\beta \to 0 \\ \mu = 0}]{} -\frac{4e^2}{\pi^2}\frac{c(0,0)}{\beta^2} = \frac{e^2 T^2}{3}.$$

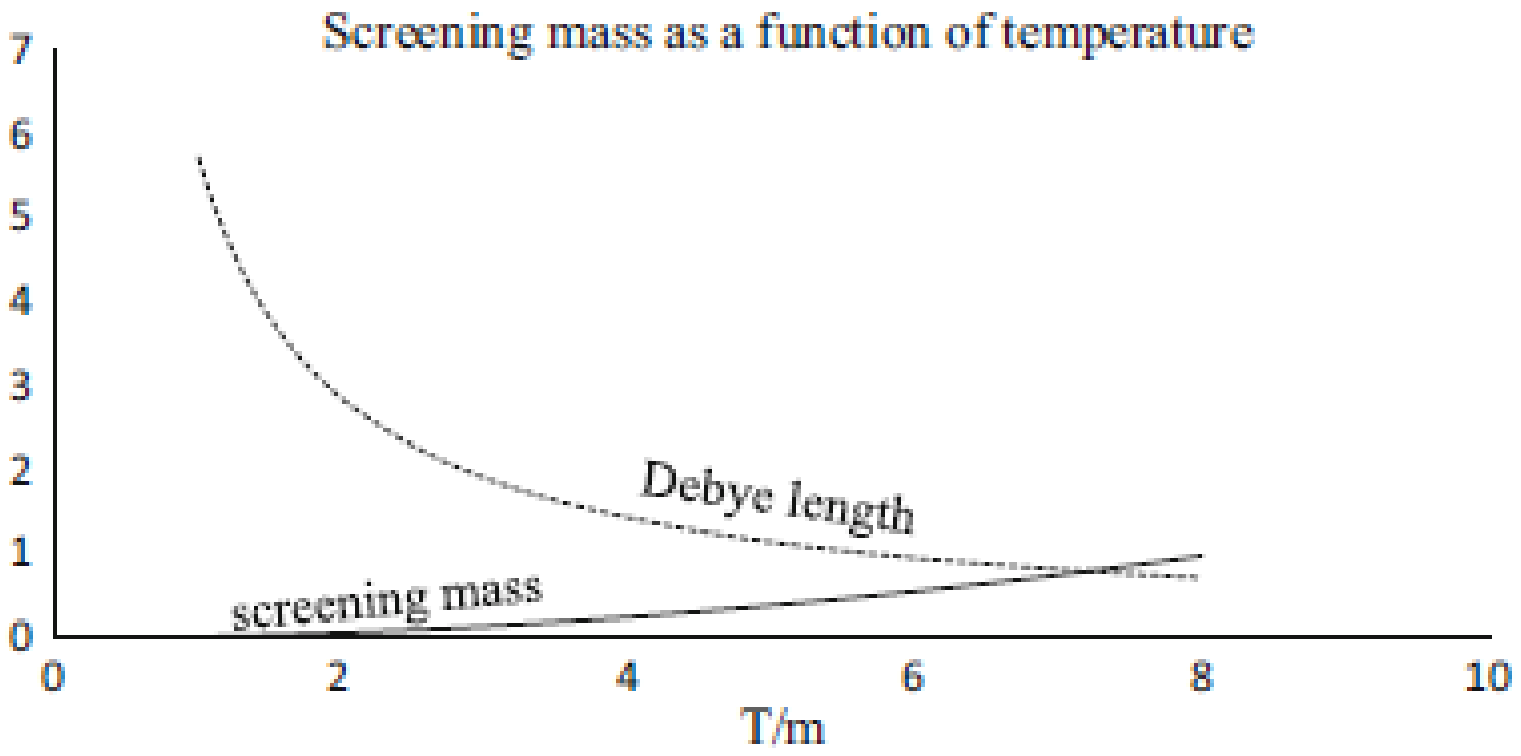
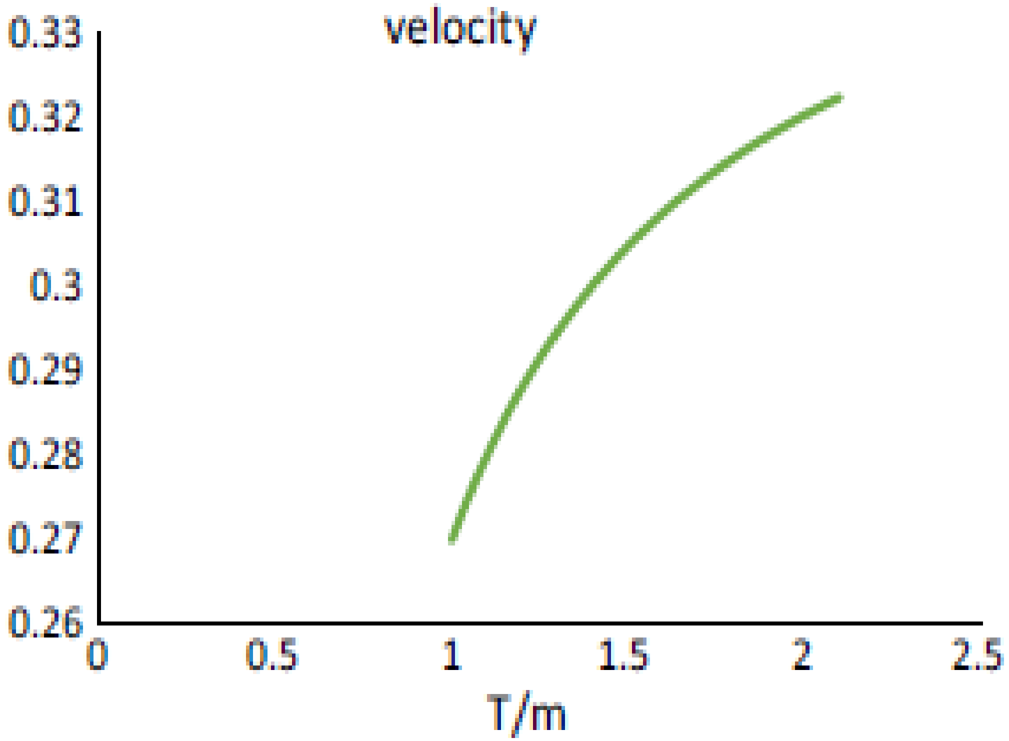
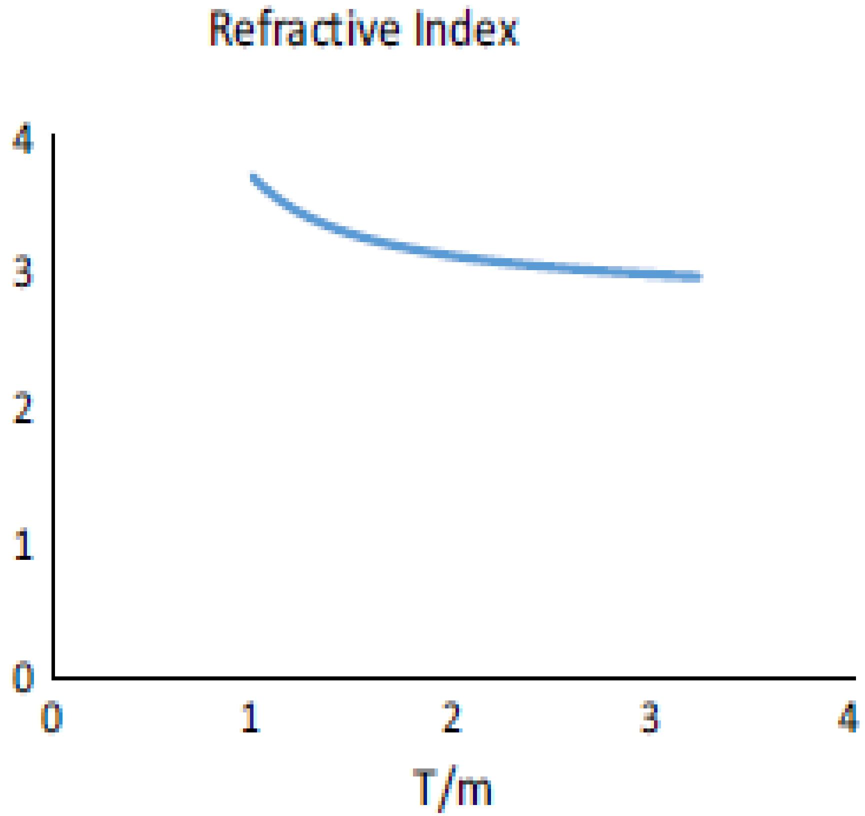


Fig. 5 Plot of plasma screening mass (solid line) and the corresponding Debye shielding length (broken line) as a function of temperature in units of the electron mass


velocity
0.33
0.32
0.31
0.3
0.29
0.28
0.27
0.26
0
0.5
1
1.5
2
2.5
T/m


Refractive Index
4
3
2
1
0
0
1
2
3
4
T/m

Electromagnetic Properties

Electric Permittivity

Magnetic Permeability

$$\varepsilon(k) = 1 - \frac{\pi_L(K)}{k^2}, \qquad \frac{1}{\mu(k)} = \frac{k^2\pi_T(K) - \omega^2\pi_L(K)}{k^2K^2},$$

Speed

Refractive Index

$$\varepsilon(K) = 1 + \chi_e$$

$$\mu(K) = 1 + \chi_m$$

$$v_{\text{prop}} = \sqrt{\frac{1}{\varepsilon(K)\mu(K)}}, \qquad r_i = \frac{c}{v} = \sqrt{\frac{\varepsilon(K)\mu(K)}{\varepsilon_0(K)\mu_0(K)}}.$$

χ_e and χ_m give the dielectric constant and magnetization of a medium

(1) For $\mu \gg T$ and $\omega \gg m \gg k$, we obtain from the above equations

$$\varepsilon_E(k,\mu) = 1 + \frac{2\alpha}{\pi^2}\left[\frac{\mu^2}{4k^2} + \frac{2m^2\omega^2}{k^4}\ln\frac{\mu}{m}\right],$$

$$\frac{1}{\mu_B(k,\mu)} = 1 + \frac{2\alpha}{\pi}\frac{\omega^6}{k^6}\left[\frac{3}{16}\frac{k^2}{4m^2} - \frac{2m^2}{\omega^2}\ln\frac{\mu}{m}\right].$$

(2) For $\mu \gg T$ and $\omega \ll m \ll k$,

$$\varepsilon_E(k,\mu) = 1 + \frac{\alpha}{2\pi^2}\left[\frac{\mu^2}{k^2} + \frac{11}{48}\frac{k^2}{m^2}\right],$$

$$\frac{1}{\mu_B(k,\mu,T)} = 1 + \frac{\alpha}{2\pi}\left[\frac{3}{32}\frac{k^2}{m^2}\right].$$

$$\varepsilon(K) \cong 1-\frac{4e^2}{\pi^2 K^2}\left(1-\frac{\omega^2}{\mathbf{k}^2}\right)\left\{\left(1-\frac{\omega}{2\mathbf{k}}\ln\frac{\omega+\mathbf{k}}{\omega-\mathbf{k}}\right)\left(\frac{ma(m\beta,\mu)}{\beta}-\frac{c(m\beta,\mu)}{\beta^2}\right)\right.$$
$$\left.+\frac{1}{4}\left(2m^2-\omega^2+\frac{11k^2+37\omega^2}{72}\right)b(m\beta,\mu)\right\},$$

$$\frac{1}{\mu(K)} \cong 1-\frac{2e^2}{\pi^2 k^2 K^2}\left[\omega^2\left\{\left(1-\frac{\omega^2}{\mathbf{k}^4}-\left(1+\frac{\mathbf{k}^2}{\omega^2}\right)\left(1-\frac{\omega^2}{\mathbf{k}^2}\right)\frac{\omega}{2\mathbf{k}}\ln\frac{\omega+\mathbf{k}}{\omega-\mathbf{k}}\right\}\right.\right.$$
$$\left.\left.\times\left(\frac{ma(m\beta,\mu)}{\beta}-\frac{c(m\beta,\mu)}{\beta^2}\right)-\frac{1}{8}\left(6m^2-\omega^2+\frac{129\omega^2-109k^2}{72}\right)b(m\beta,\mu)\right\}\right]$$

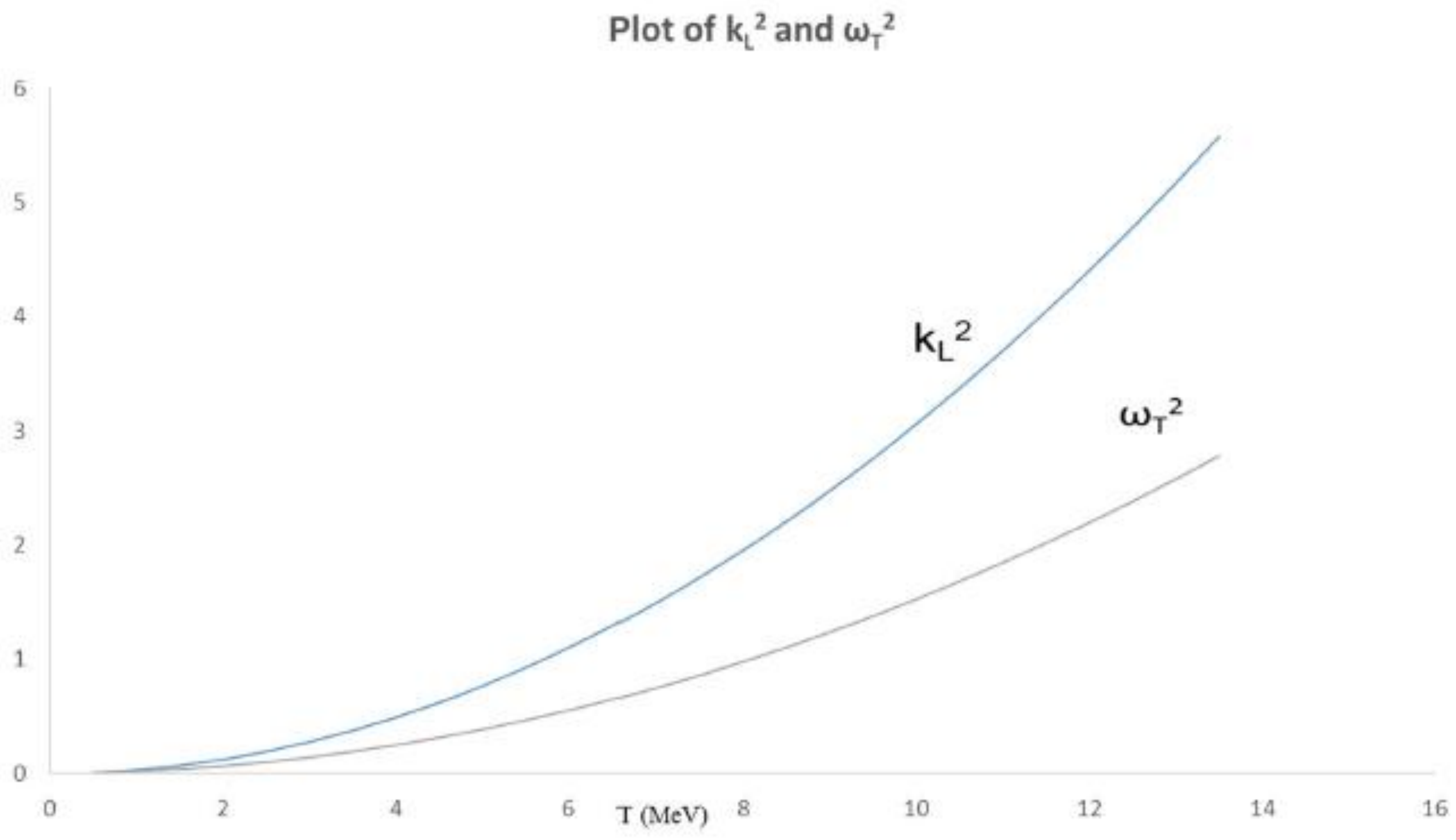


Fig. 1 Temperature dependence of the longitudinal component of the propagation vector (square) and the transverse frequency (square) is plotted showing that the Debye shielding length is greater than the plasma frequency

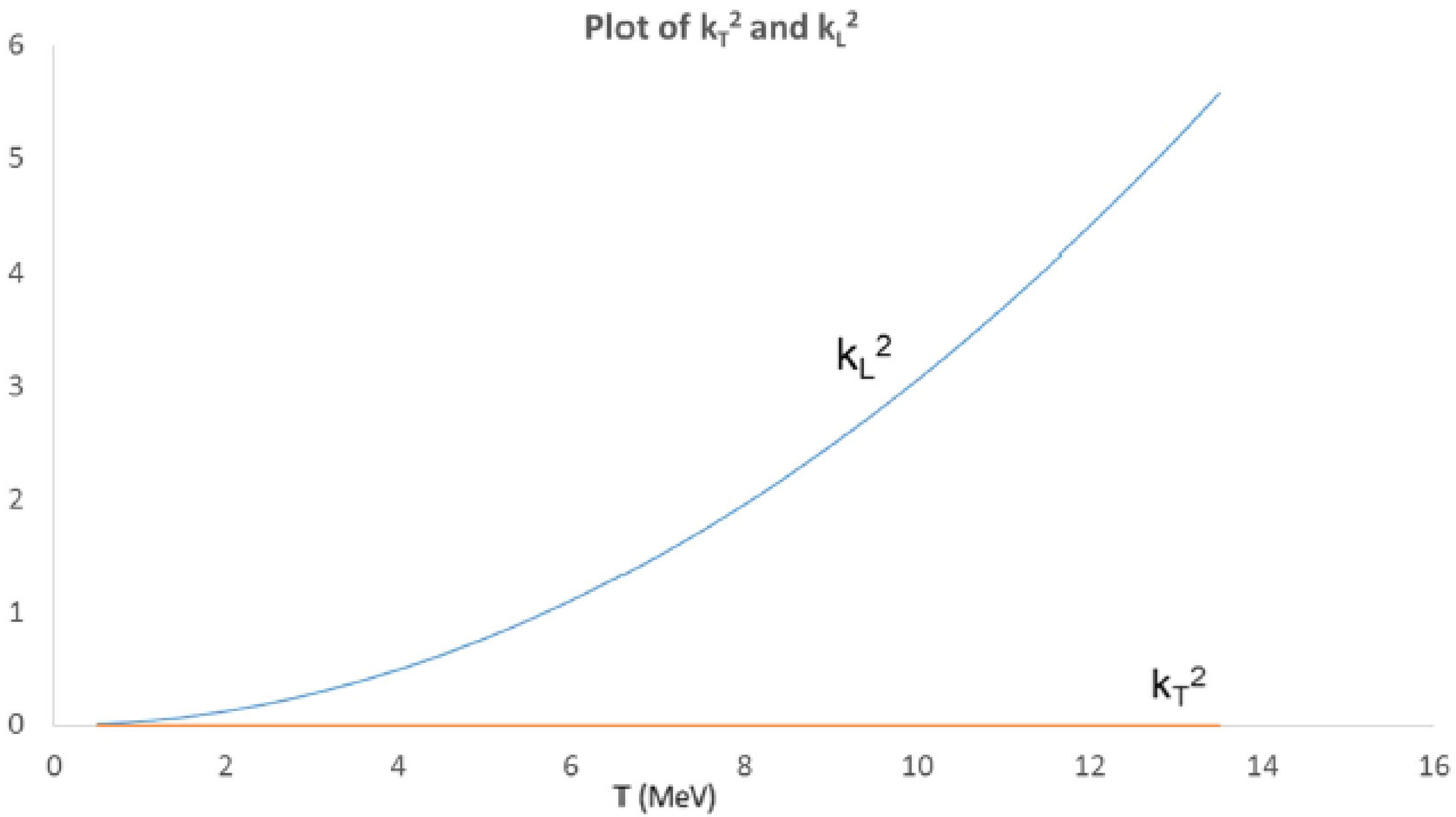


Fig. 2 On comparing the plots of K_L^2 and K_T^2, it can be clearly seen that the propagation vector in the transverse direction is independent of temperature. The inverse of Debye length $(1/k_L)$ increases with temperature, but the transverse motion will not be affected

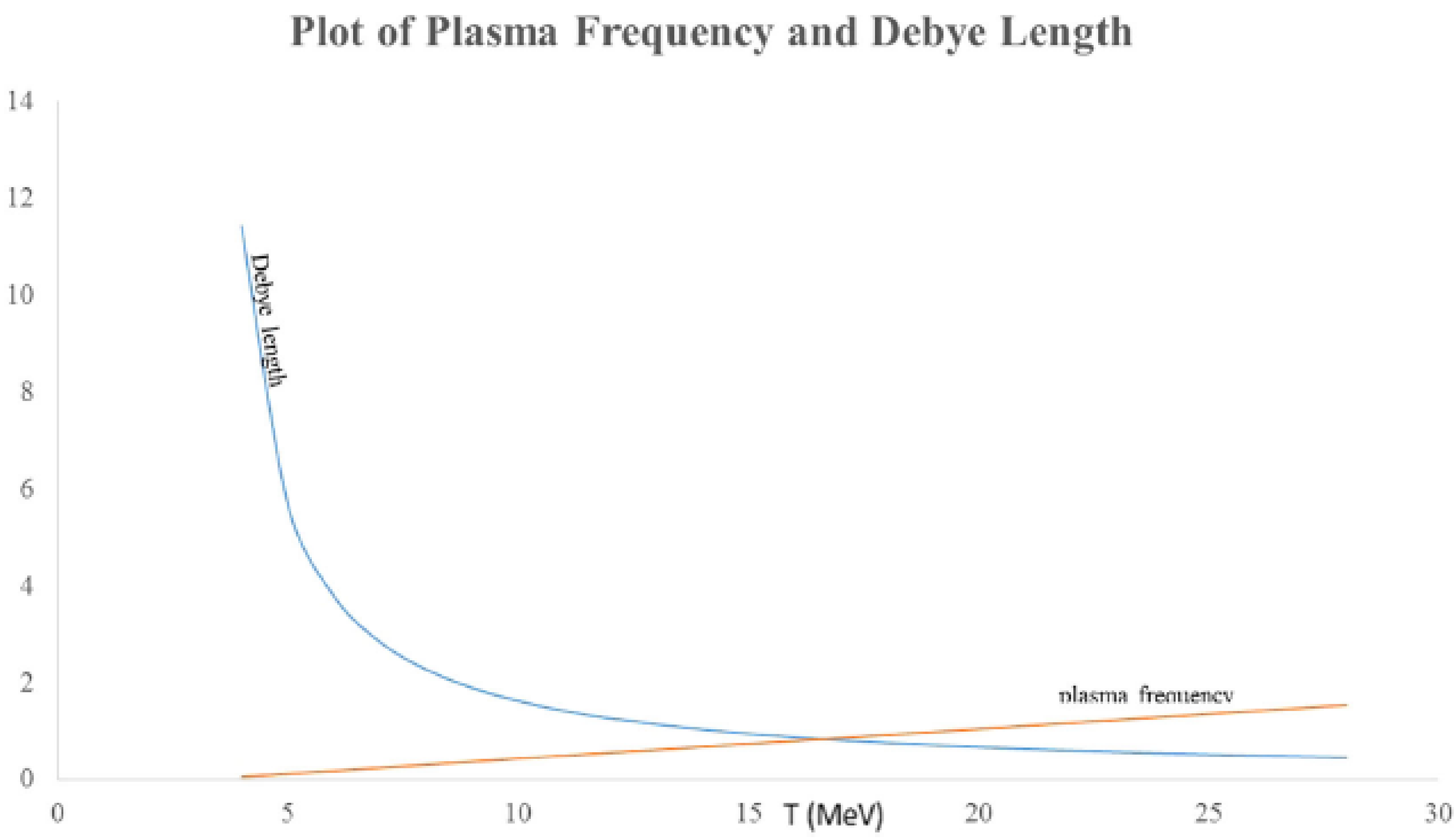


Fig. 4 Debye length and the plasma frequency are plotted as a function of temperature showing that the screening length increases with *T*. When temperature rises around 16 times the mass of the electron which is around 8 MeV, a phase transition takes place and the plasma frequency becomes greater than the shielding length

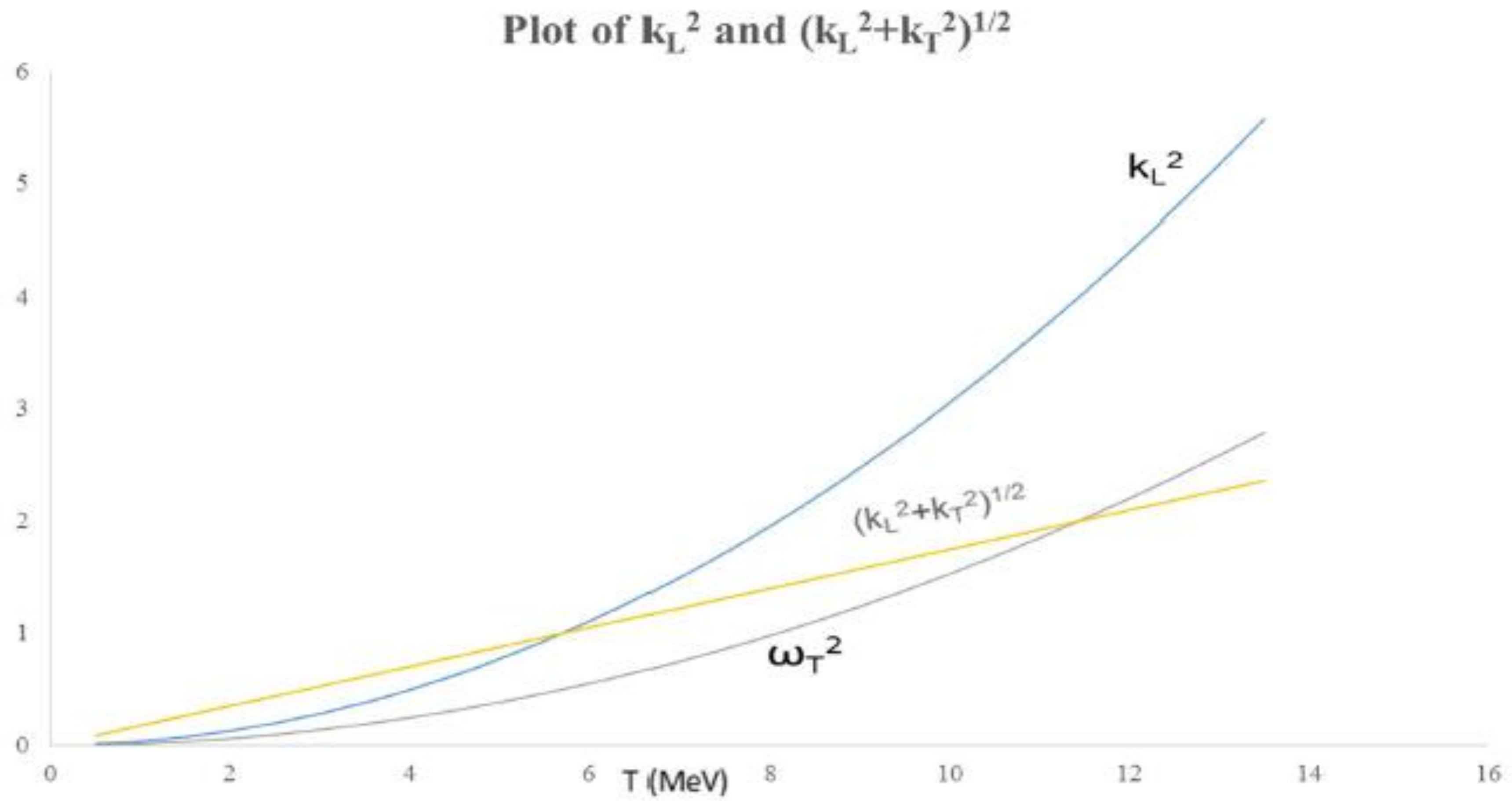


Fig. 3 Comparison of the plots of k; the magnitude of the propagation vector, longitudinal propagation K_L^2 and transverse frequency ω_T^2

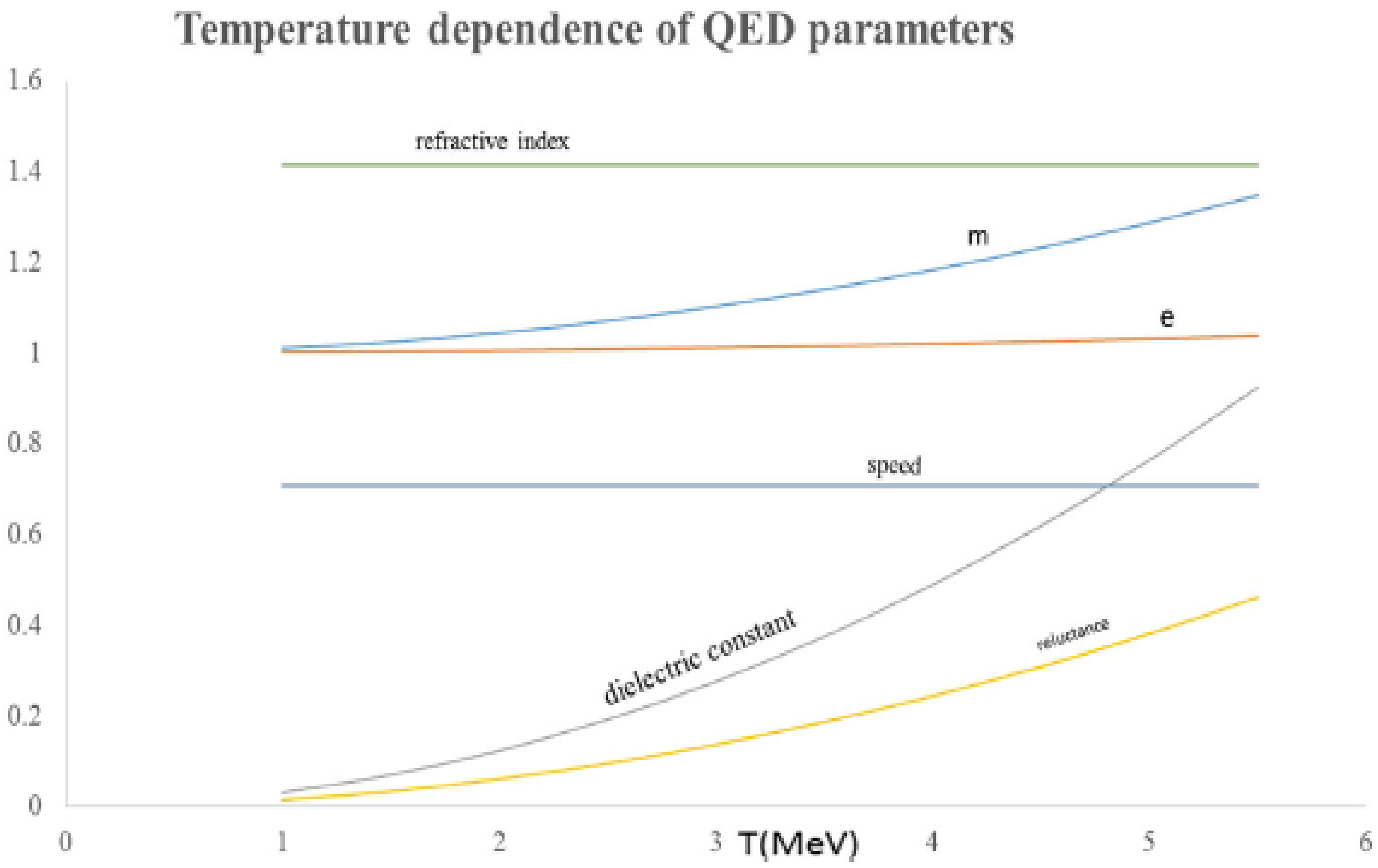


Fig. 5 All of the plasma parameters are plotted as functions of temperature such that the reluctance, propagation speed, and dielectric constant all increase with temperature

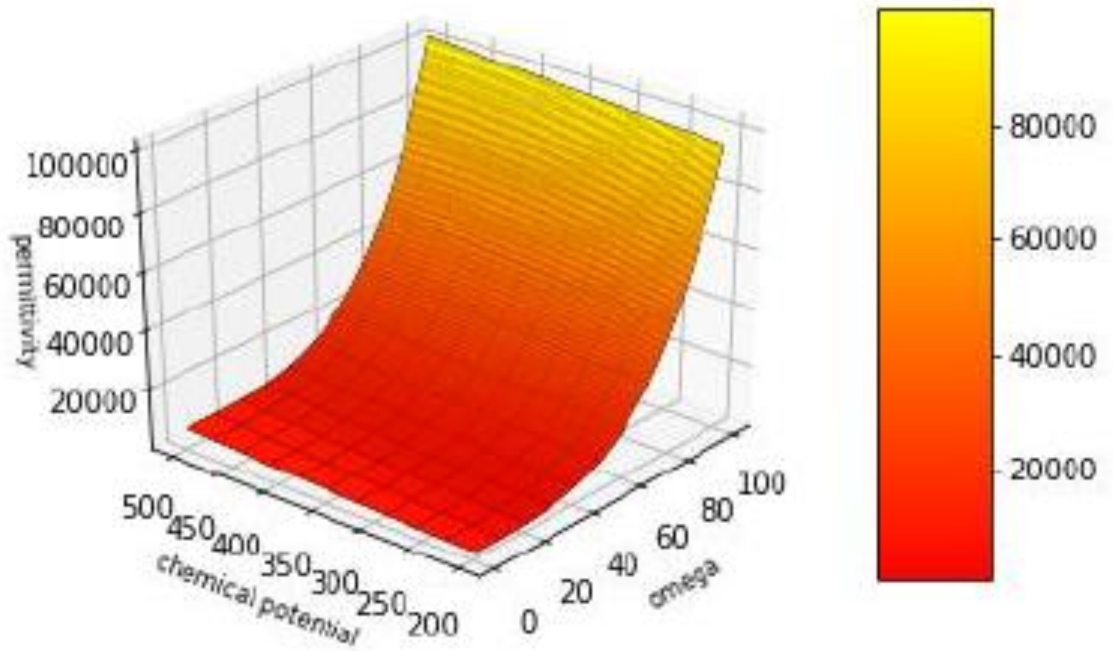


Figure 1: Relative permittivity ϵ_R is plotted versus omega ω in units of electron mass $\mathbf{m_e}$ for constant k

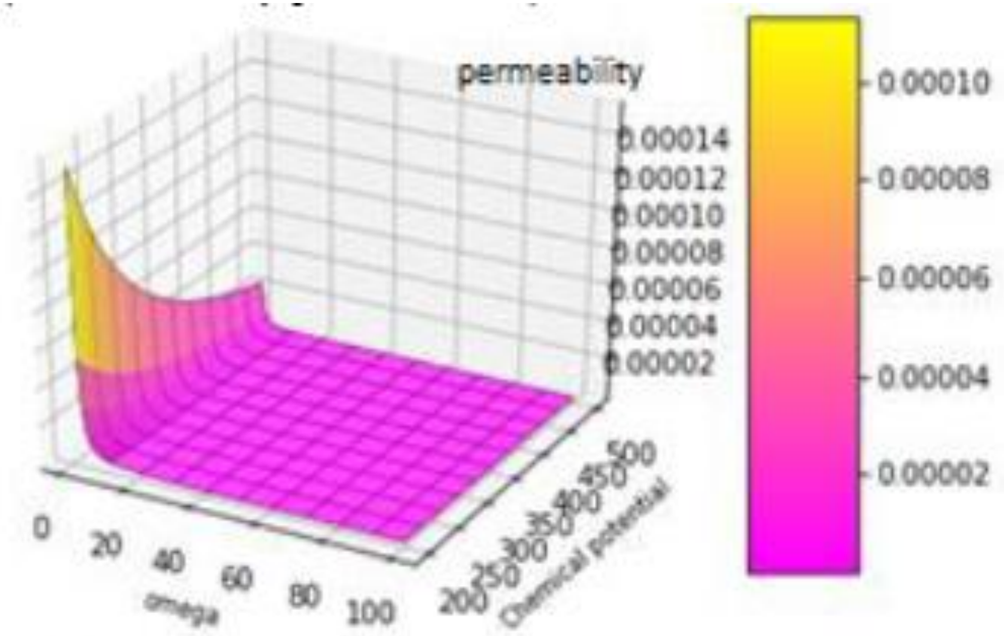


Figure 2: Relative permeability $\boldsymbol{\mu_B}$ is plotted versus omega ω in units of electron mass $\mathbf{m_e}$ for constant k

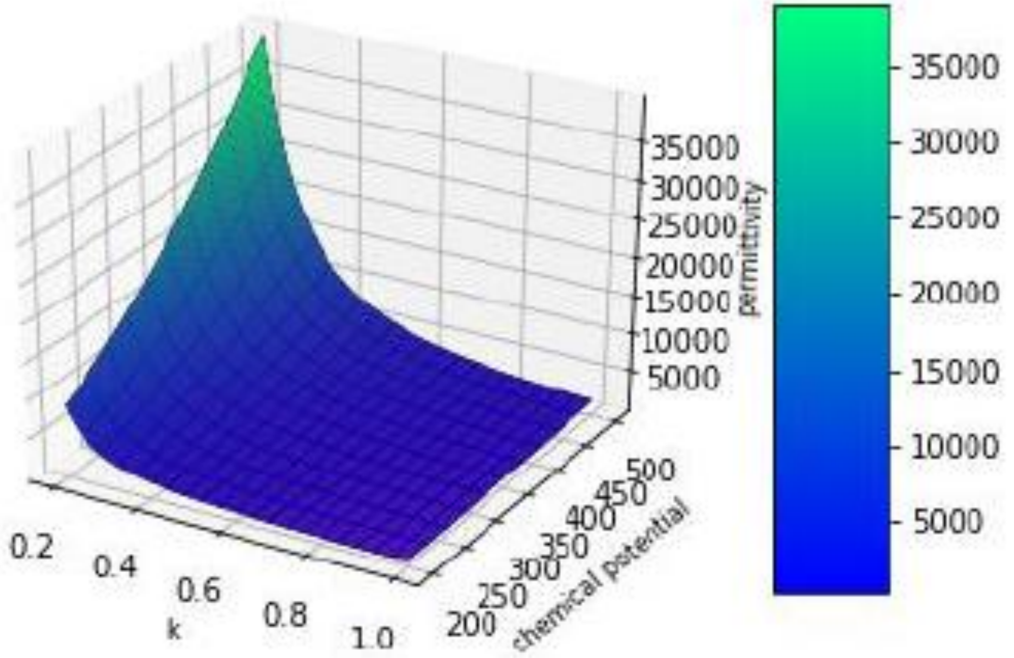


Figure 3: Relative permittivity ϵ_R is plotted versus k in units of electron mass $\mathbf{m_e}$ for constant ω

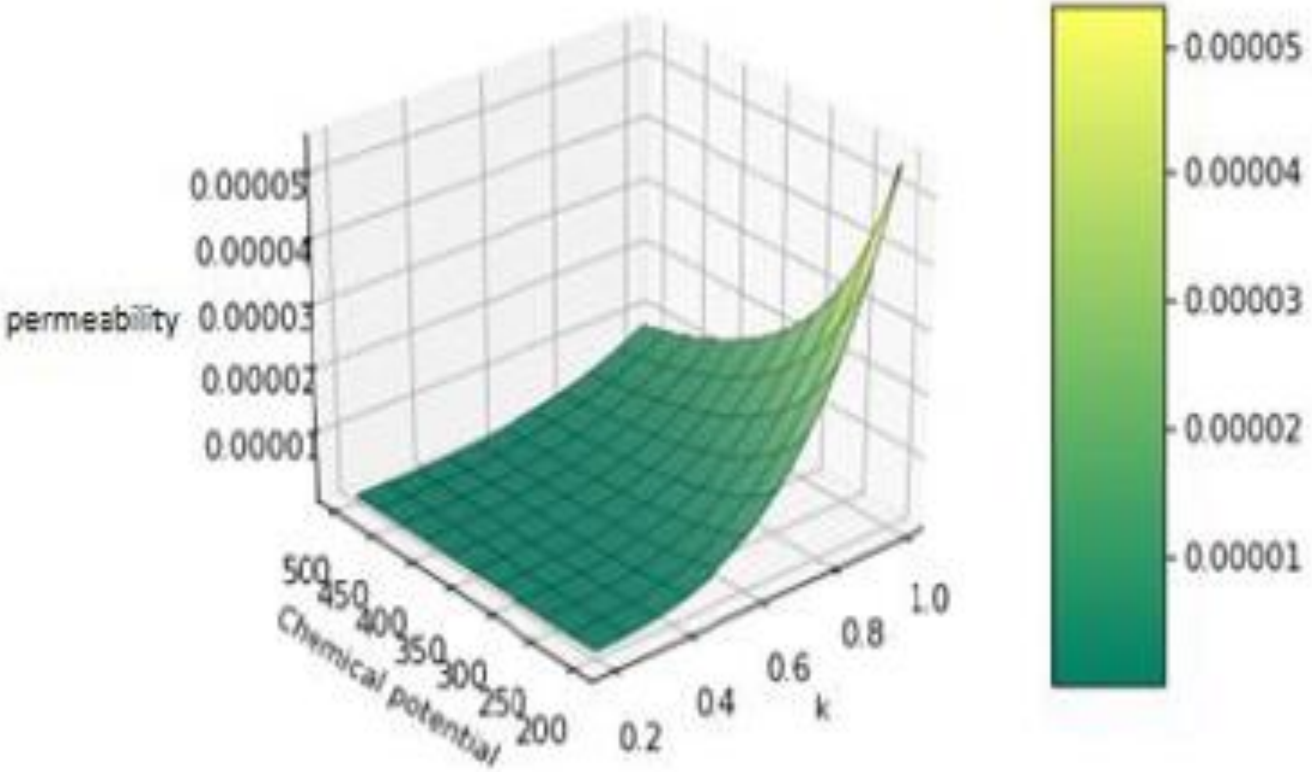


Figure 4: Relative permeability μ_B is plotted versus k in units of electron mass m_e for constant ω

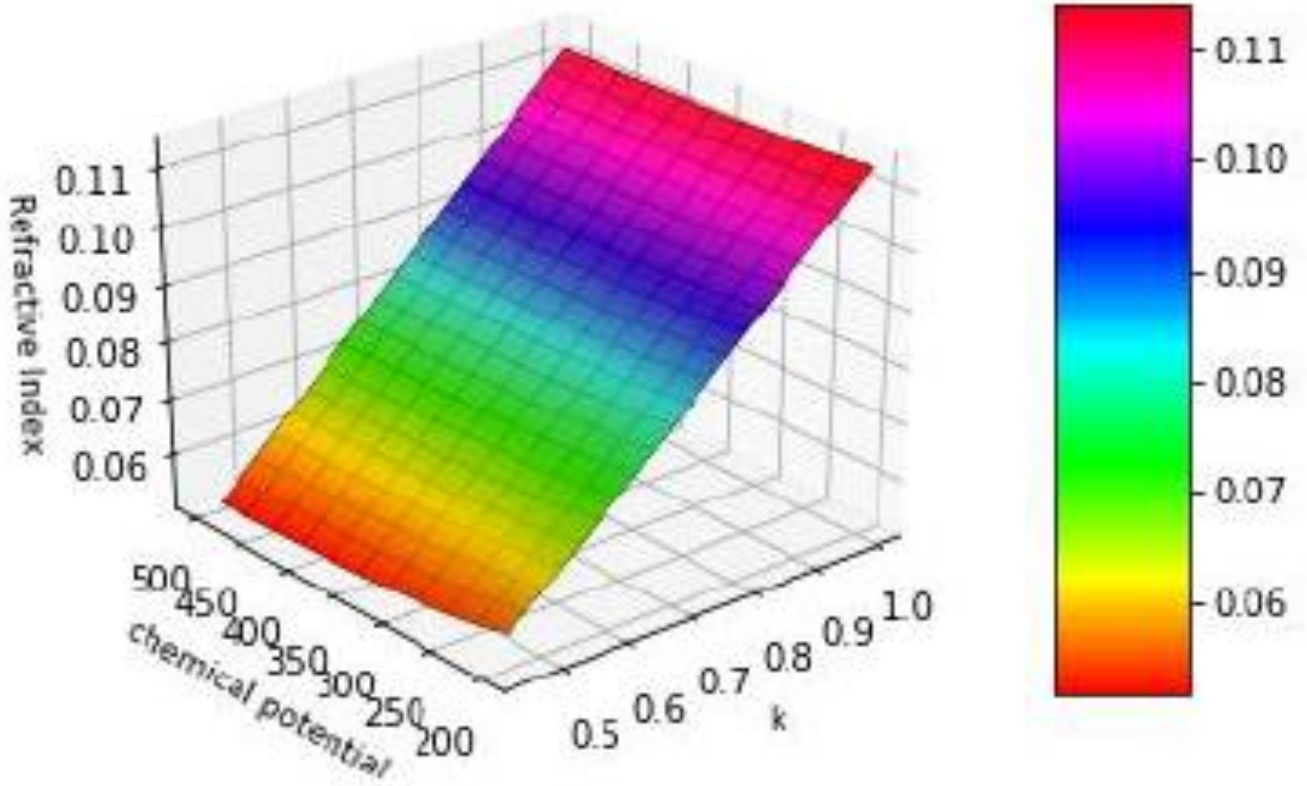


Figure 6: Index of refraction is plotted versus k in units of electron mass $\mathbf{m_e}$ for constant ω

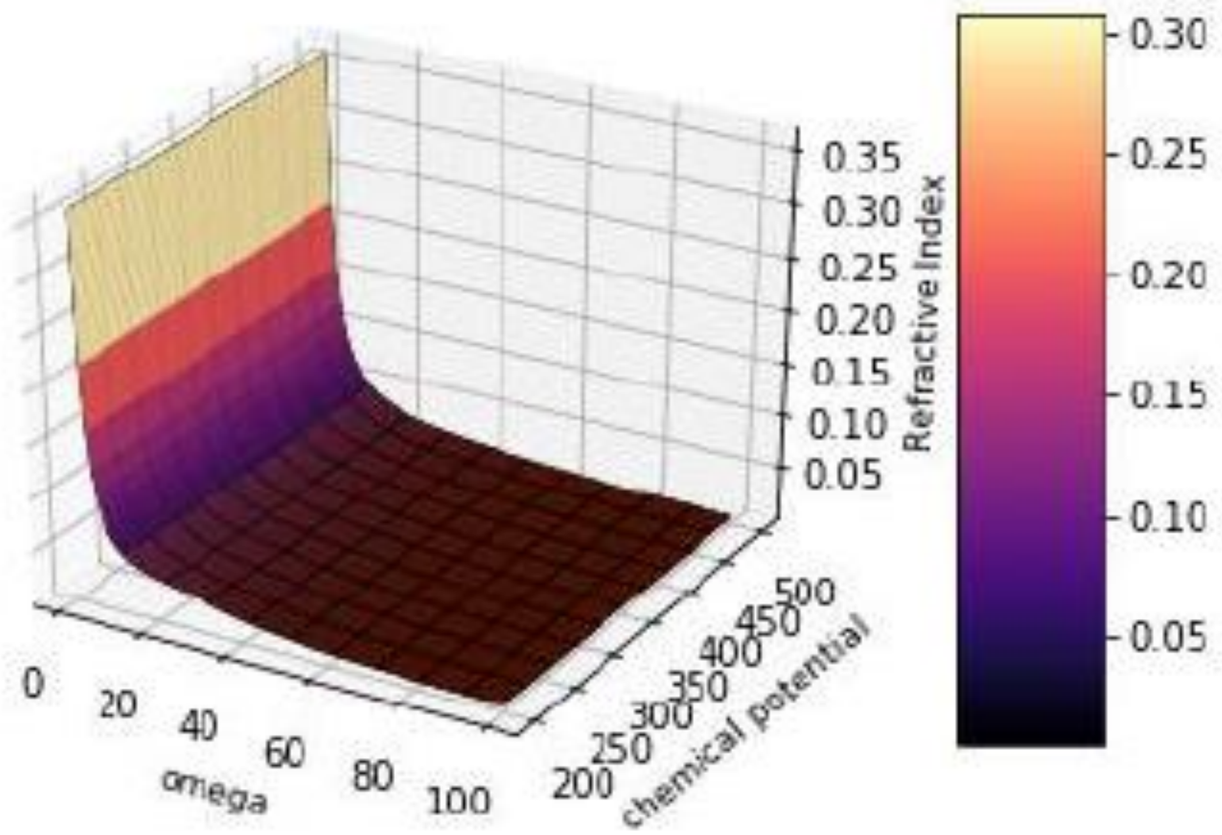
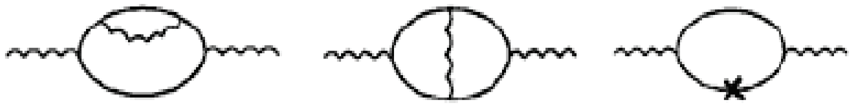


Figure 5: Index of refraction is plotted versus omega (ω) in units of electron mass $\mathbf{m_e}$ for constant k

Second order Vacuum Polarization



(a) (b) (c)

$$
{}^2\Pi^a_{\mu\nu}(p) = e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \times \mathrm{Tr}\left[\gamma_\mu S_\beta(k)\gamma_\rho D^{\rho\sigma}_\beta(q-k) S_\beta(q)\gamma_\sigma S_\beta(k)\gamma_\nu S_\beta(p-k)\right],
$$

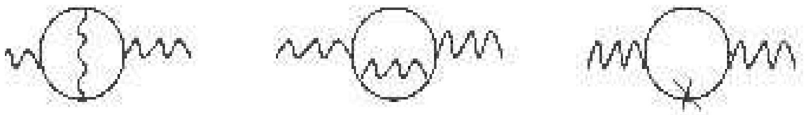
$$
{}^2\Pi^b_{\mu\nu}(p) = e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \times \mathrm{Tr}\left[\gamma_\mu S_\beta(k)\gamma_\rho D^{\rho\sigma}_\beta(q-k) S_\beta(q)\gamma_\nu S_\beta(q-p)\gamma_\sigma S_\beta(k-p)\right].
$$

$$
P_{\mu\nu} = (g_{\mu\nu} - u_\mu u_\nu) + \frac{1}{|\mathbf{p}|^2}(p_\mu - \omega u_\mu)(p_\nu - \omega u_\nu) \qquad Q_{\mu\nu} = \frac{1}{p^2|\mathbf{p}|^2}\left[\{p^2 u_\mu + \omega(p_\mu - \omega u_\mu)\}\{p^2 u_\nu + \omega(p_\nu - \omega u_\nu)\}\right]
$$

Electromagnetic Properties of a Thermal Medium

$$\omega_L^2 \longrightarrow \lim_{|\mathbf{p}|\longrightarrow 0} \Pi_L(|\mathbf{p}|, |\mathbf{p}|, T) = 0,$$

$$\omega_T^2 \longrightarrow \lim_{|\mathbf{p}|\longrightarrow 0} \Pi_T(|\mathbf{p}|, |\mathbf{p}|, T) = \frac{\alpha^2 T^2}{6}.$$



(a) (b) (c)

QED coupling constant

$$\alpha_R = \alpha(T=0)\left(1 + \frac{\alpha^2 T^2}{6m^2}\right).$$

Electric Permittivity

$$\begin{aligned} \varepsilon(p,T) &= 1 - |\mathbf{p}|^2 \Pi_L(p,T) \\ &= 1 - \frac{2\alpha^2 T^2 p^2}{3}\left(1 + \frac{p_0^2}{2m^2}\right), \end{aligned}$$

Magnetic Permeability

$$\begin{aligned} \frac{1}{\mu(p,T)} &= 1 + \frac{1}{p^2}\left[\Pi_T(p,T) - \frac{p_0^2}{|\mathbf{p}|^2}\Pi_L(p,T)\right] \\ &= 1 + \frac{\alpha^2 T^2}{3}\left[\frac{1}{2p^2} - \frac{1}{|\mathbf{p}|^2}\left(1 + \frac{p_0^2}{2m^2}\right)\left(1 + \frac{2p_0^2}{|\mathbf{p}|^2}\right)\right]. \end{aligned}$$

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Electromagnetic Properties of Hot and Dense Media

For Temperatures smaller than the Chemical Potential

$$\varepsilon(K,\mu)=1-\frac{2\alpha}{\pi K^{3}}\left[\mu^{2}\omega\left[1-\frac{m^{2}}{\mu^{2}}\right]\ln\frac{\omega-k}{\omega+K}-A_{i}\ln\frac{\mu}{m}+\frac{1}{2m^{2}}\left[\frac{m^{2}}{\mu^{2}}-1\right]A_{2}\right]$$

$$\frac{1}{\mu(K)}=1+\frac{2\alpha}{\pi kK^{2}}\left\{\frac{1}{4}\left[1-\frac{m^{2}}{\mu^{2}}\right]\left[\frac{k\mu^{2}}{2}+\frac{k^{3}}{12}\left[1-\frac{3\omega^{2}+k^{2}}{4m^{2}}\right]+\frac{5m^{2}\omega^{2}k}{2k^{2}}+\frac{3m^{2}k}{8}-\omega\mu^{2}\left[1-\frac{m^{2}}{\mu^{2}}\right]\left[1-\frac{\omega^{2}}{k^{2}}\right]\left[\frac{\omega^{2}}{k^{2}}+\frac{1}{4}\right]\ln\frac{\omega-k}{\omega+k}-\frac{\omega^{2}}{k^{2}}\left[1-\frac{\omega^{2}}{k^{2}}\right]-\frac{1}{2}\left[\frac{\omega^{2}}{k^{2}}+1\right]A_{1}\ln\frac{\mu}{m}+\frac{A_{2}}{4m^{2}}\left[1-\frac{m^{2}}{\mu^{2}}\right]\left[1+\frac{2\omega^{2}}{k^{2}}\right]\left[1-\frac{\omega^{2}}{k^{2}}\right]\right]\right\}.$$

$$T \ll 4\pi\sigma \ll \frac{1}{a}.$$

$$\alpha \equiv 2a(4\pi\sigma)$$

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