ELECRIC CHARGE IN STARS

M.I. Krivoruchenko
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Lectures plan:

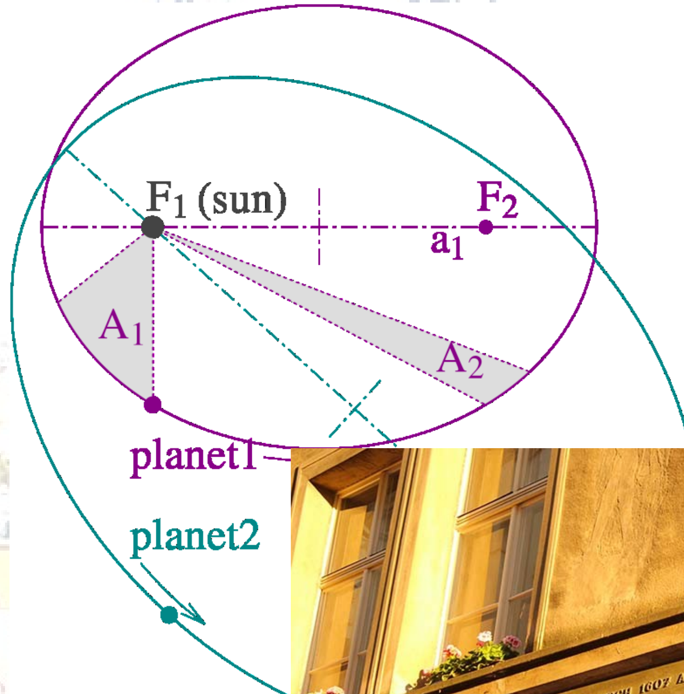
1. The formation, life and death of stars.
2. Rosseland's paper (1924) with a particular solution of hydrostatic equilibrium for a self gravitating two-component charged fluid.
3. Singular differential equations. Poincare's theorem on analyticity. Dyson's argument.
4. General solution.
5. Relativistic theory of self gravitating multicomponent charged fluid.
6. Maximum electric charge of galaxy.
7. Open questions.

Astroneutrino Theory Workshop
Czech Technical University in Prague
Prague, 28 September 2021

1/2

Alphonse Mucha

Johannes Kepler



 # parameters

- **Ptolemy** 77×(2 + 2) = **154**×2 = **308**
- **Copernicus** 34×(2 + 2) = **68**×2 = **136**

  radius + period + 2 Euler angles to fix inclination of a circle to the ecliptic/deferent plane
- **Kepler** (6 + 1)×(2 + 3) + 2 = **16** + (6 + 1)×3 = **37**

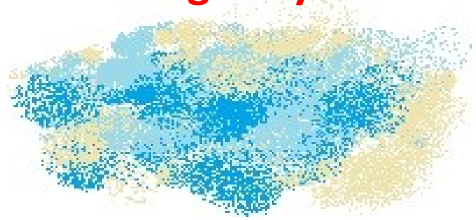
  semimajor axis + eccentricity + 3 Euler angles to fix inclination of an ellipse to the ecliptic plane

Stellar evolution

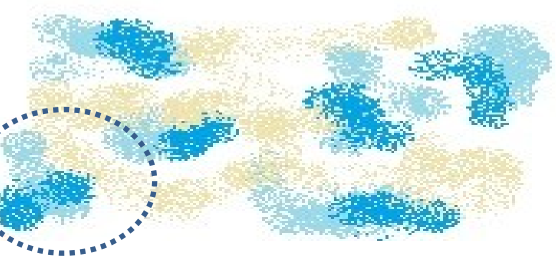
Giant Molecular Cloud in a galaxy

Gravity rips the cloud apart
as long as the substance is transparent

(a)

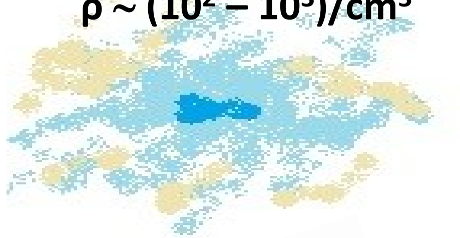


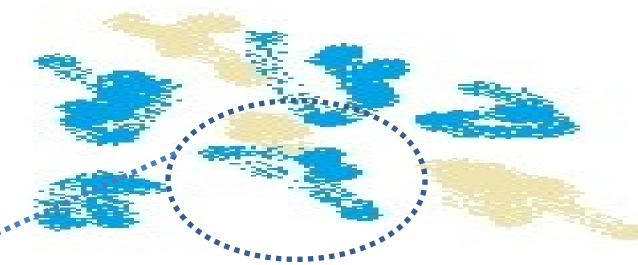
Jeans instability



R ~ (10 – 10^3) light-years
M ~ (10^4 – 10^7)$M_\odot$ & ***T*** ~ 10 – 30 K
ρ ~ (10^2 – 10^3)/cm³

(b)

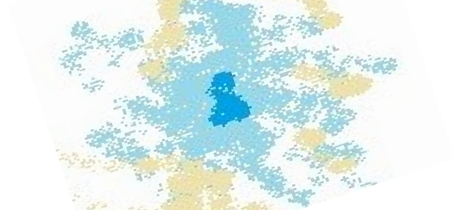




……

Protostar

(z)



R > R$_\odot$***, M ~ M***$_\odot$***, T***$_{core}$ ~ 10^4 K
center is like a black body

NB: $R_{\text{Milky Way}}$ ~ 10^5 light-years
$M_{\text{Milky Way}}$ ~ 10^{12} $M_\odot$
$T_\odot^{\text{core}}$ ~ 10^7 K

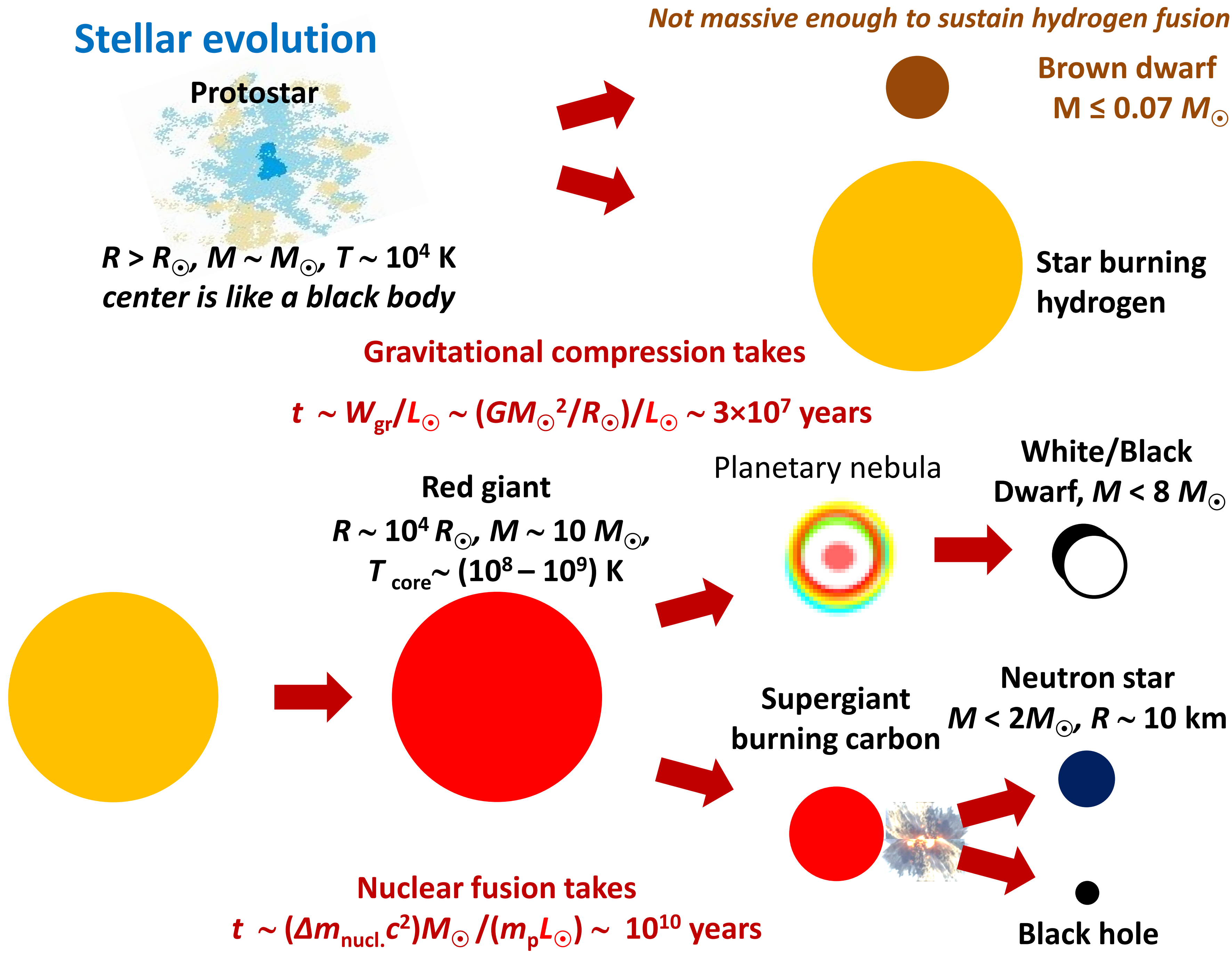
Jeans instability:

(*) $E_{\text{kinetic}} \sim \frac{M}{m_{\text{H}}} \qquad \sim \frac{GM^2}{R}$

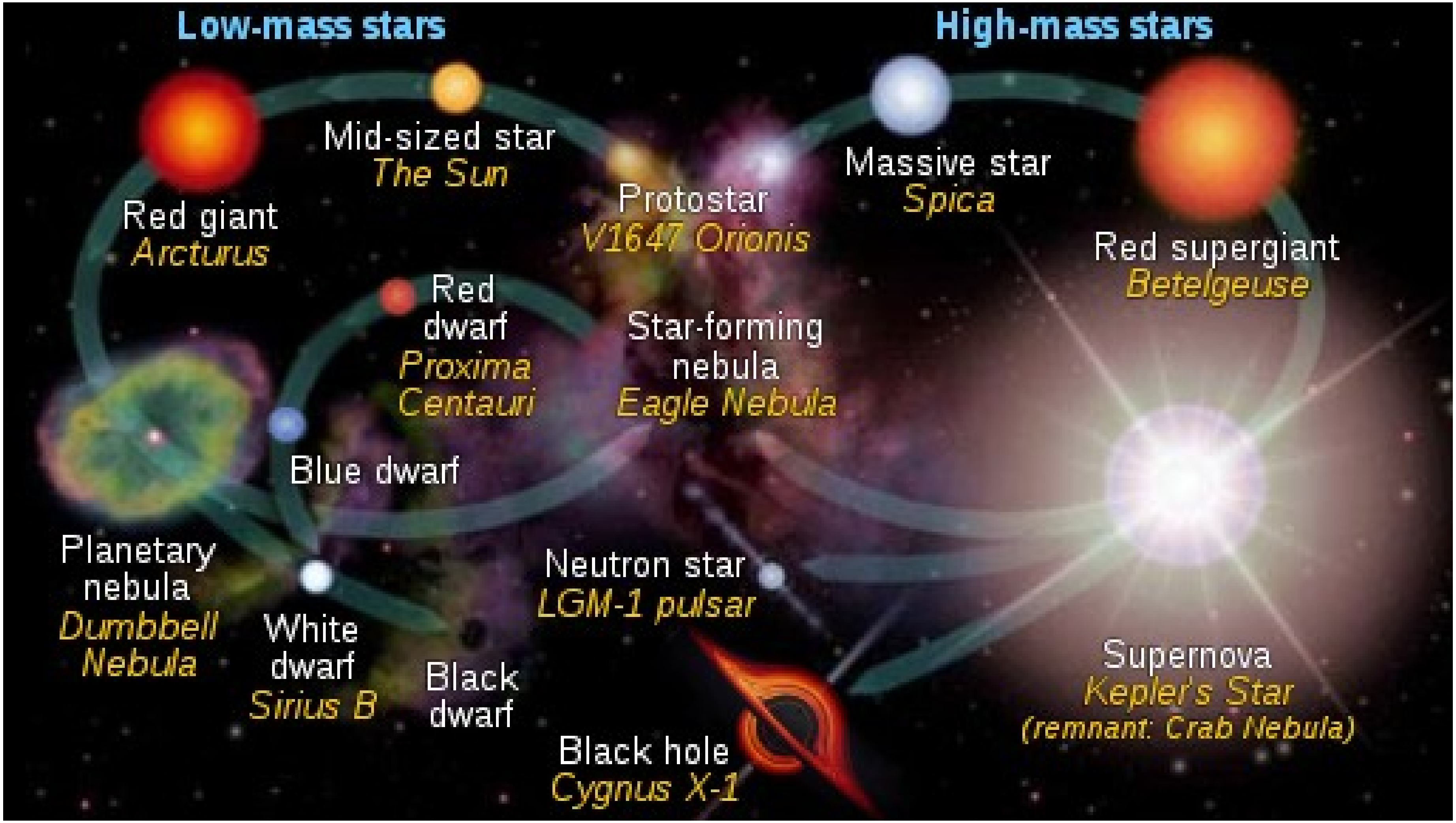
(**) $M \sim$

$$\boxed{\Rightarrow M \geq M_{\text{J}} \sim \left(\frac{kT}{Gm_{\text{H}}}\right)^{3/2} \rho^{-1/2}}$$

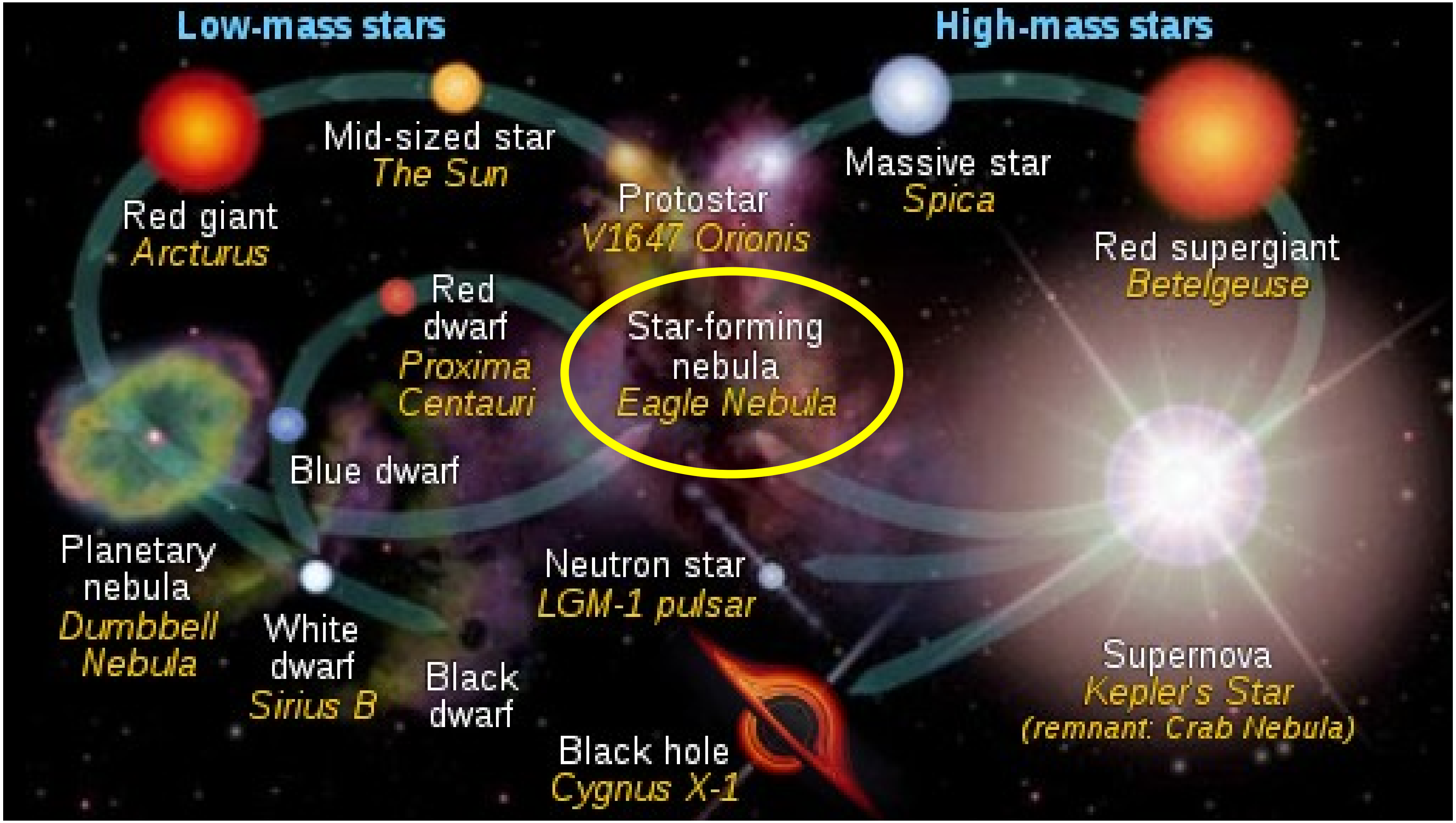
Stellar evolution



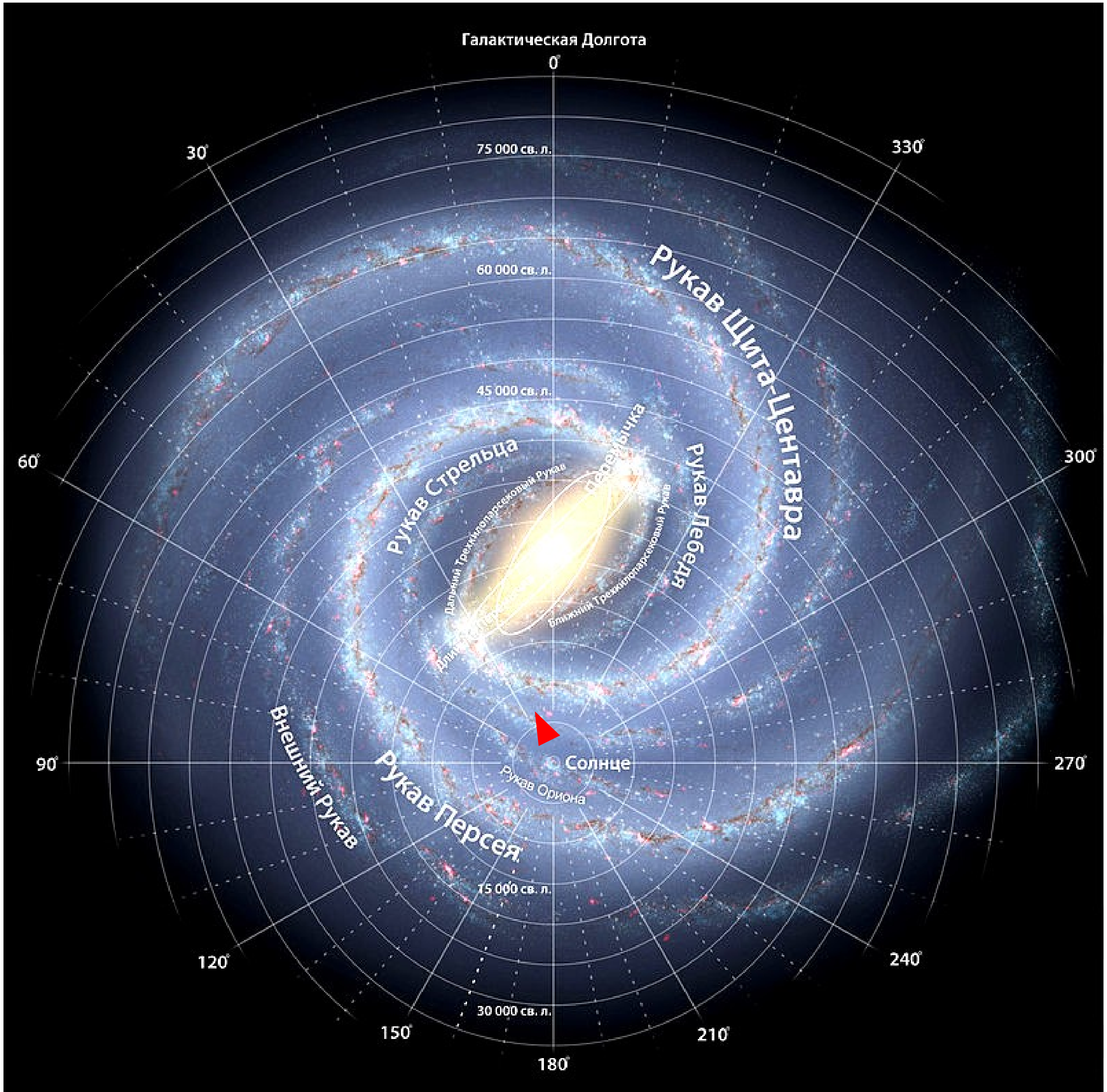
Stellar evolution



Stellar evolution

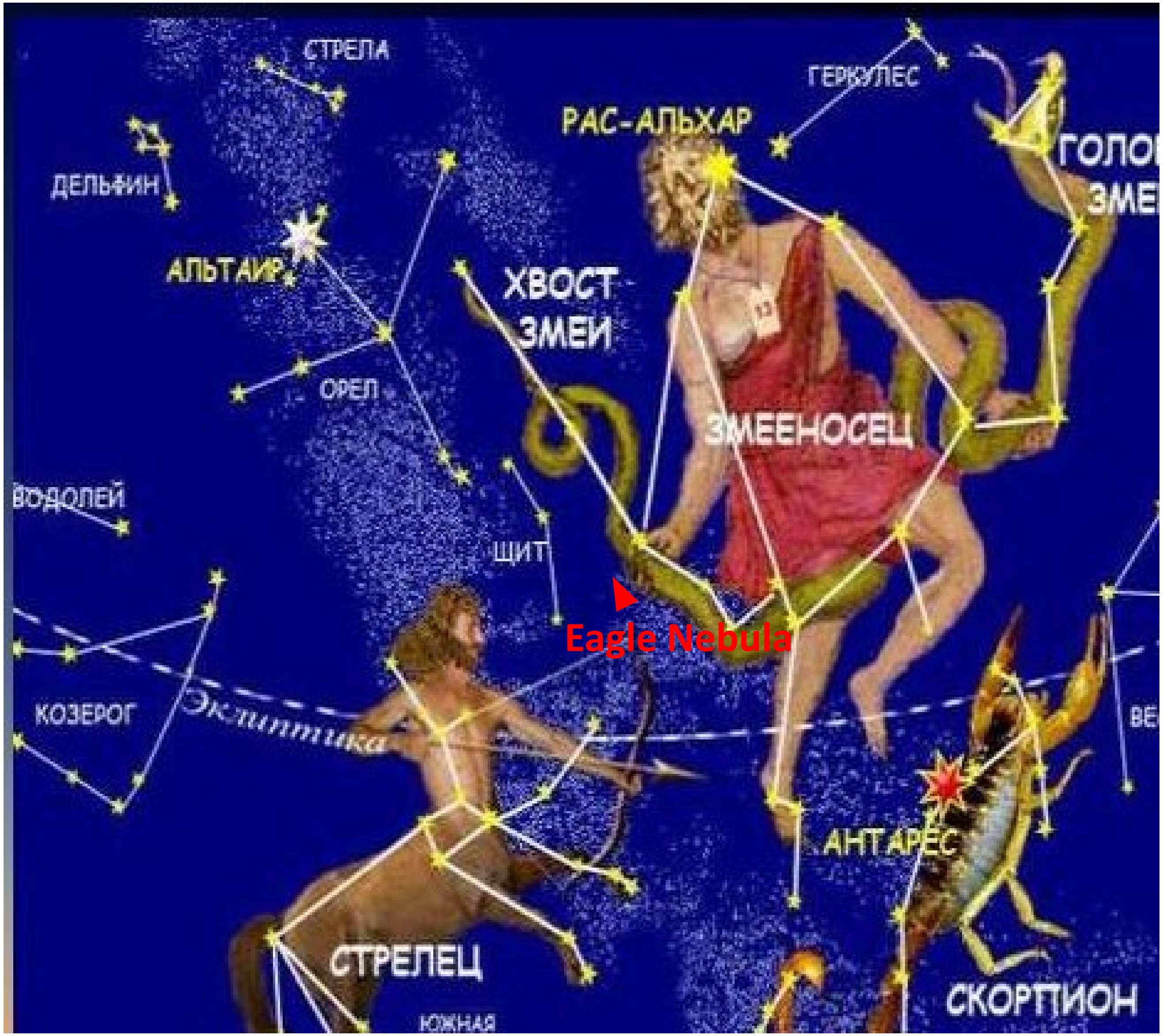


Milky Way



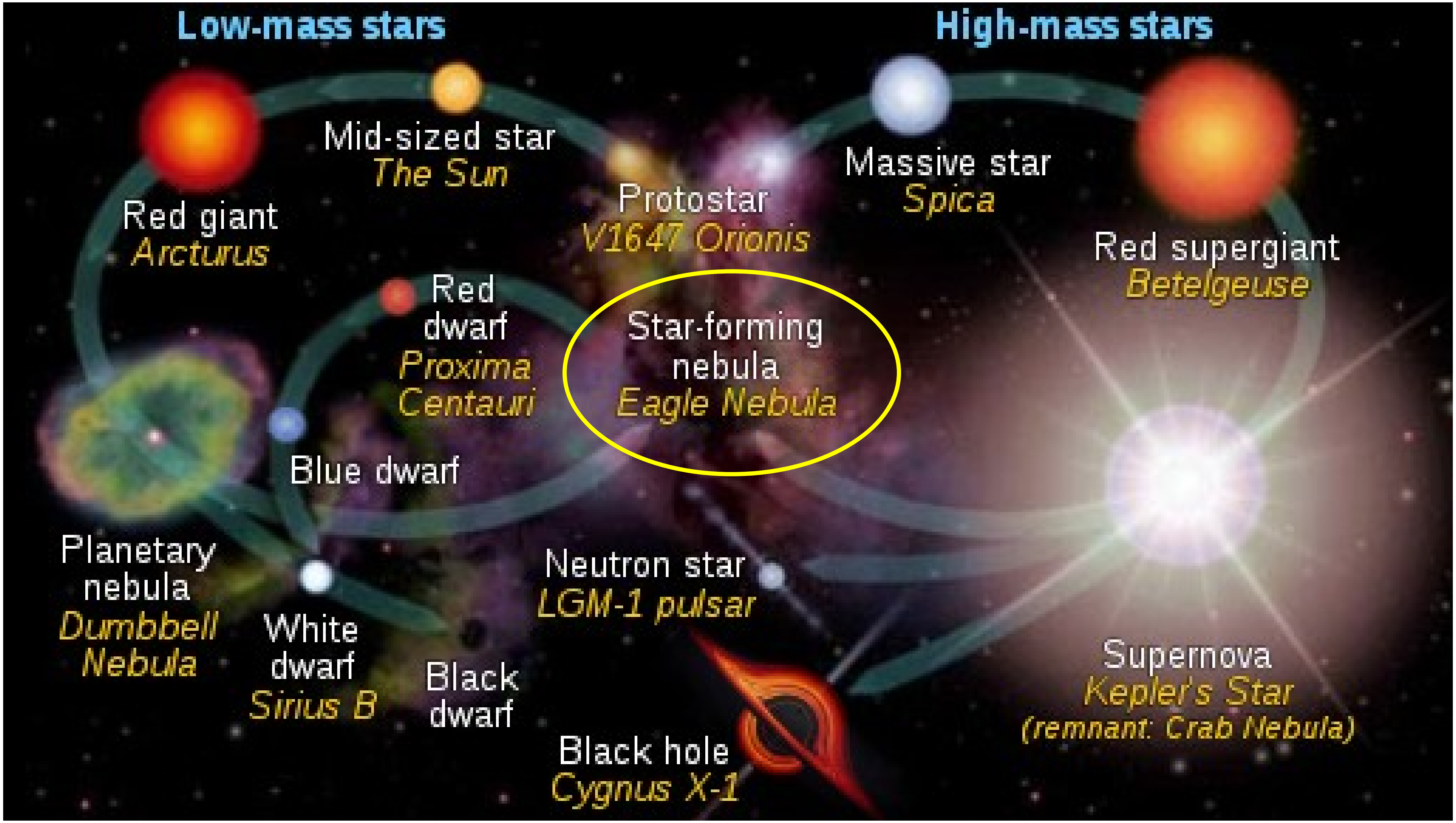
Eagle Nebula d = 5700 light-years from the Sun

Serpens constellation
one of 48 constellations described by Ptolemy
today 88 constellations are identified

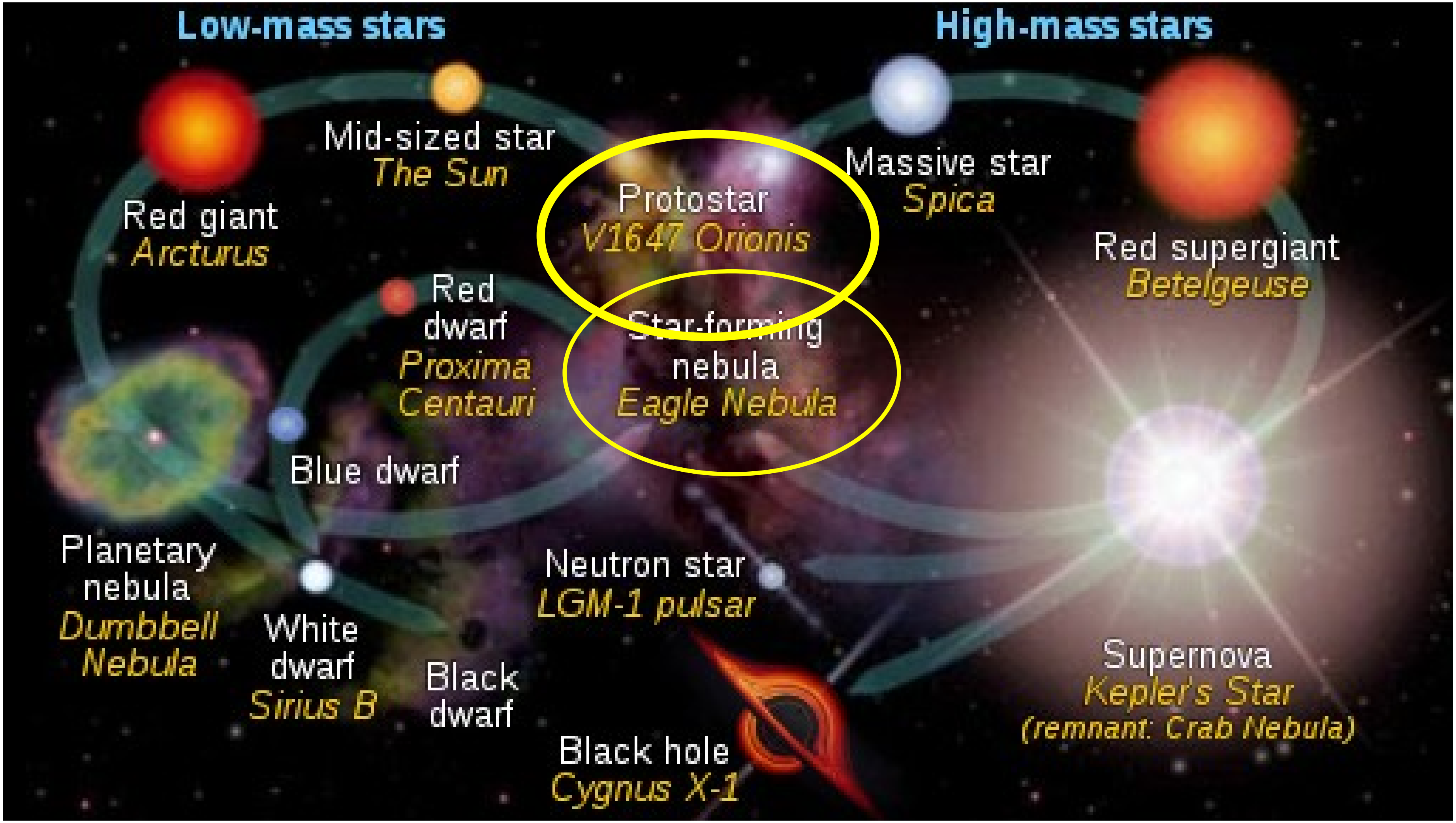


Eagle Nebula
optical image
the hydrogen gas is ionised by radiation from hot young stars

Stellar evolution



Stellar evolution



McNeil's nebula was discovered by amateur astronomer J.W. McNeil II in January 23, 2004.

New star emerges from dust cocoon

By Dr David Whitehouse
BBC News Online science editor

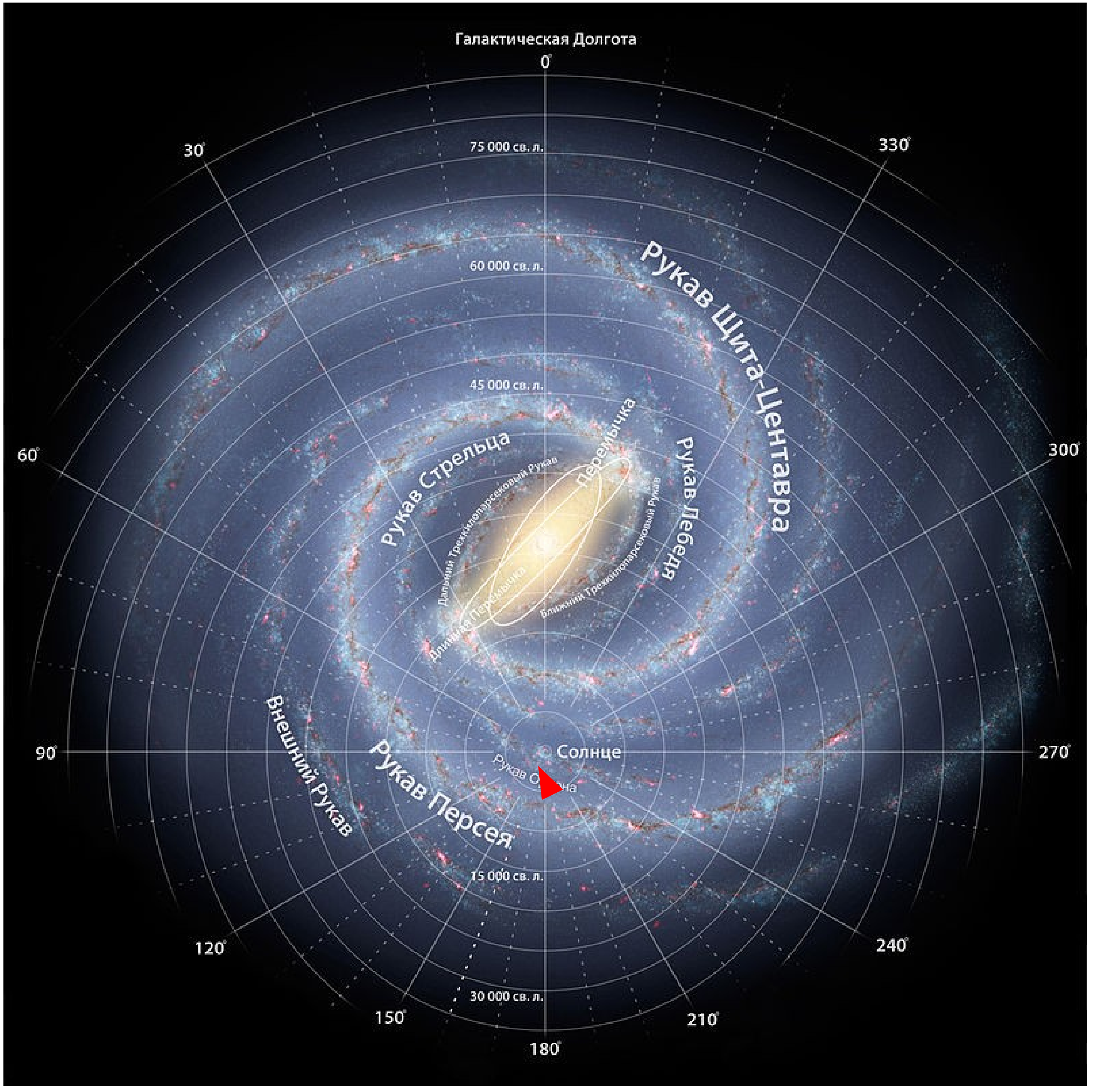
An amateur astronomer in the US has detected the emergence of a young star from the cocoon of gas and dust in which it was born.


The new object had not been observed in the gas cloud before

Such an event has only rarely been recorded by astronomers.

"This is exciting for all astronomers, especially those interested in the birth of stars," University of Hawaii astronomer Bo Reipurth told the BBC.

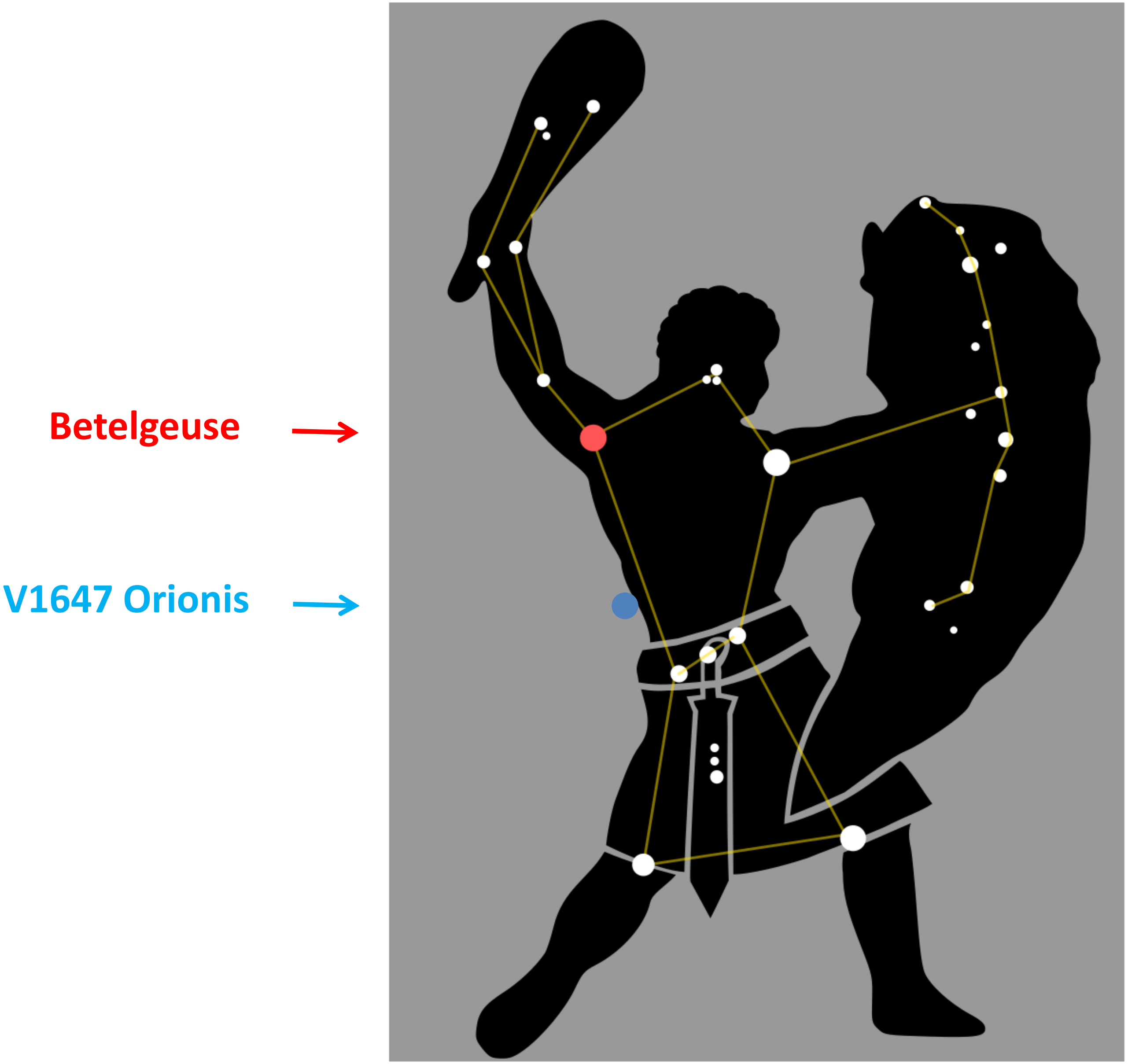
Milky Way

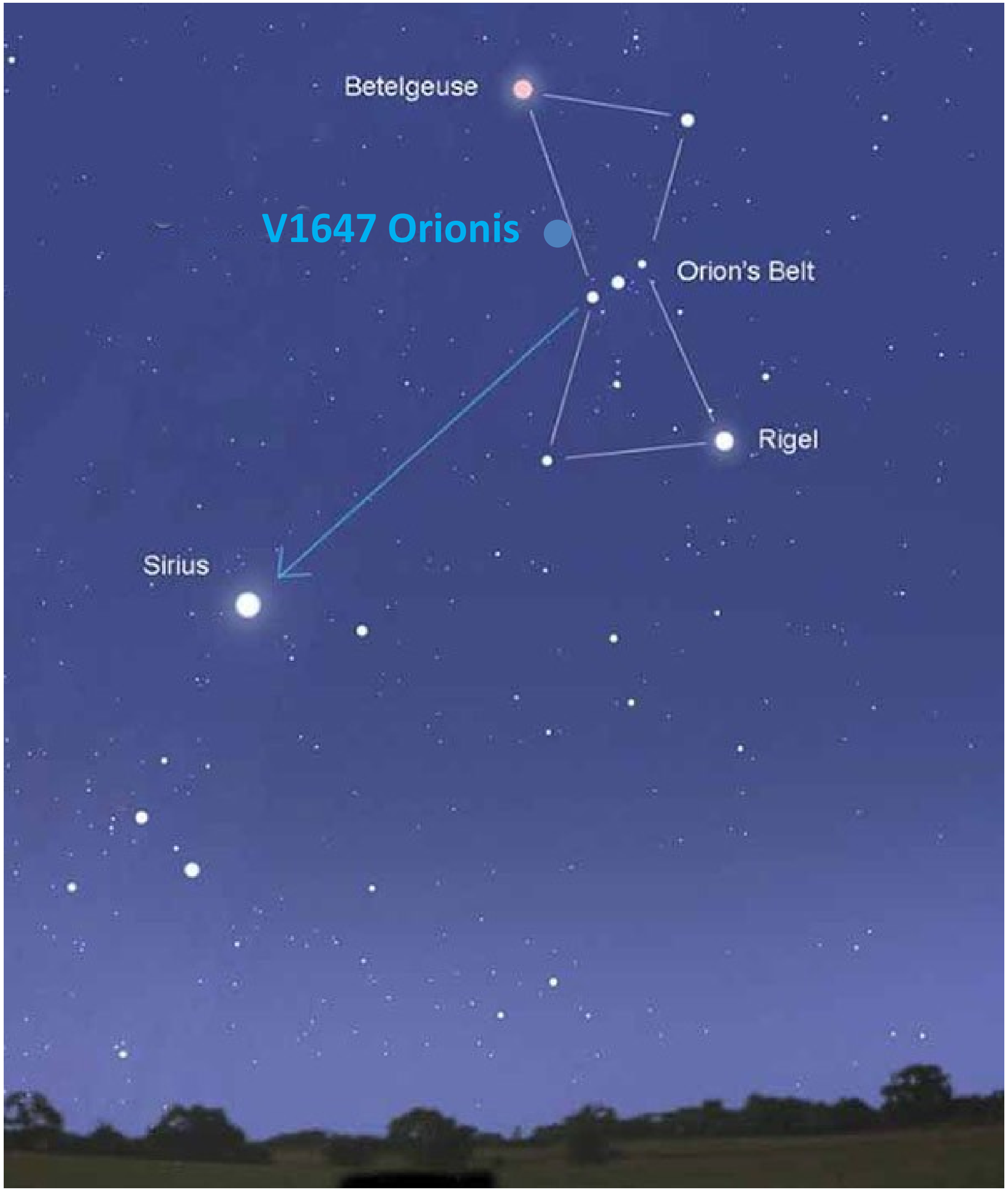


McNeil's Nebula ▶ d ~ 1500 light-years from the Sun

Orion Constellation

(named after Orion, a hunter in Greek mythology)

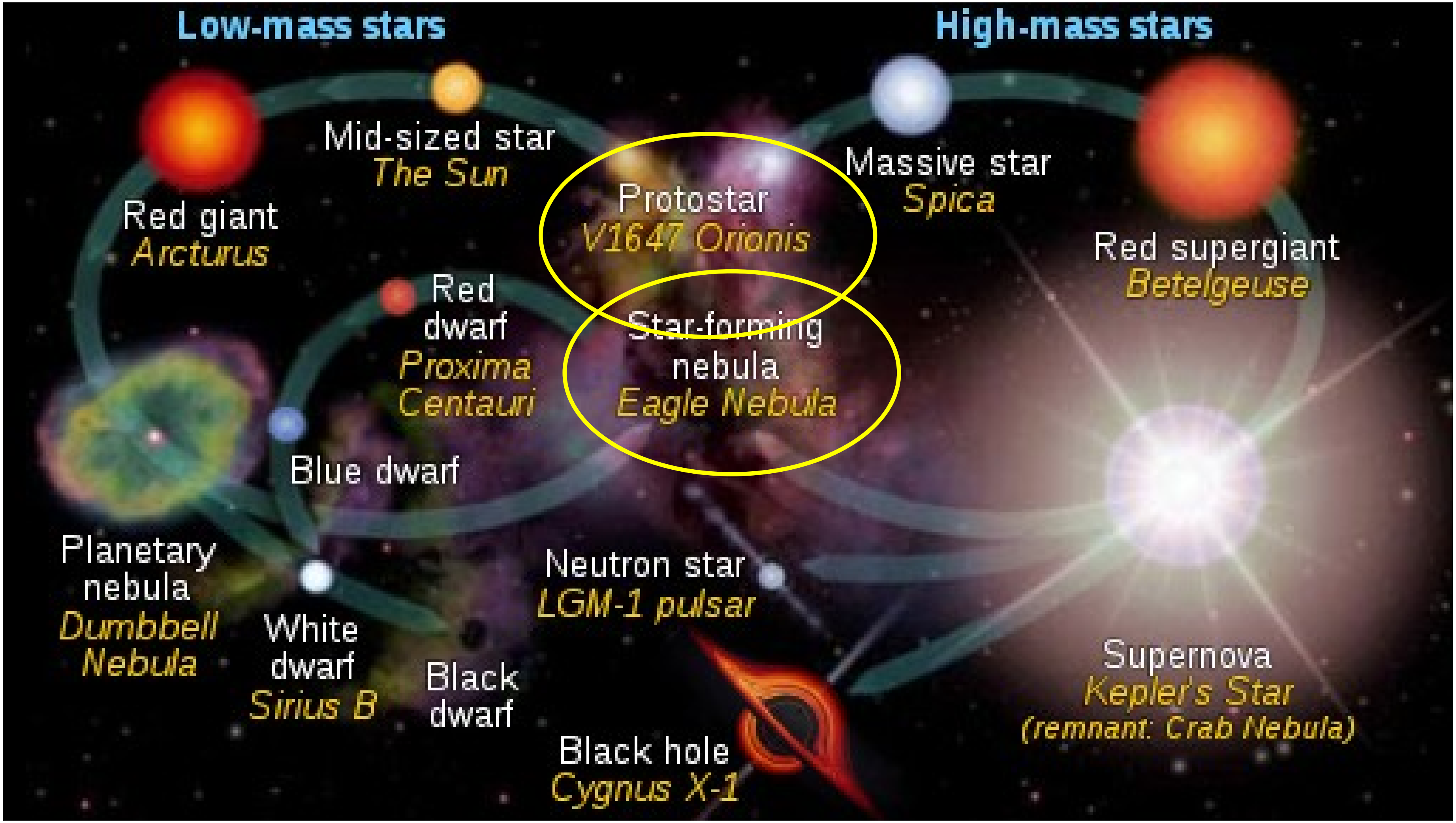



Betelgeuse
V1647 Orionis
Orion's Belt
Rigel
Sirius

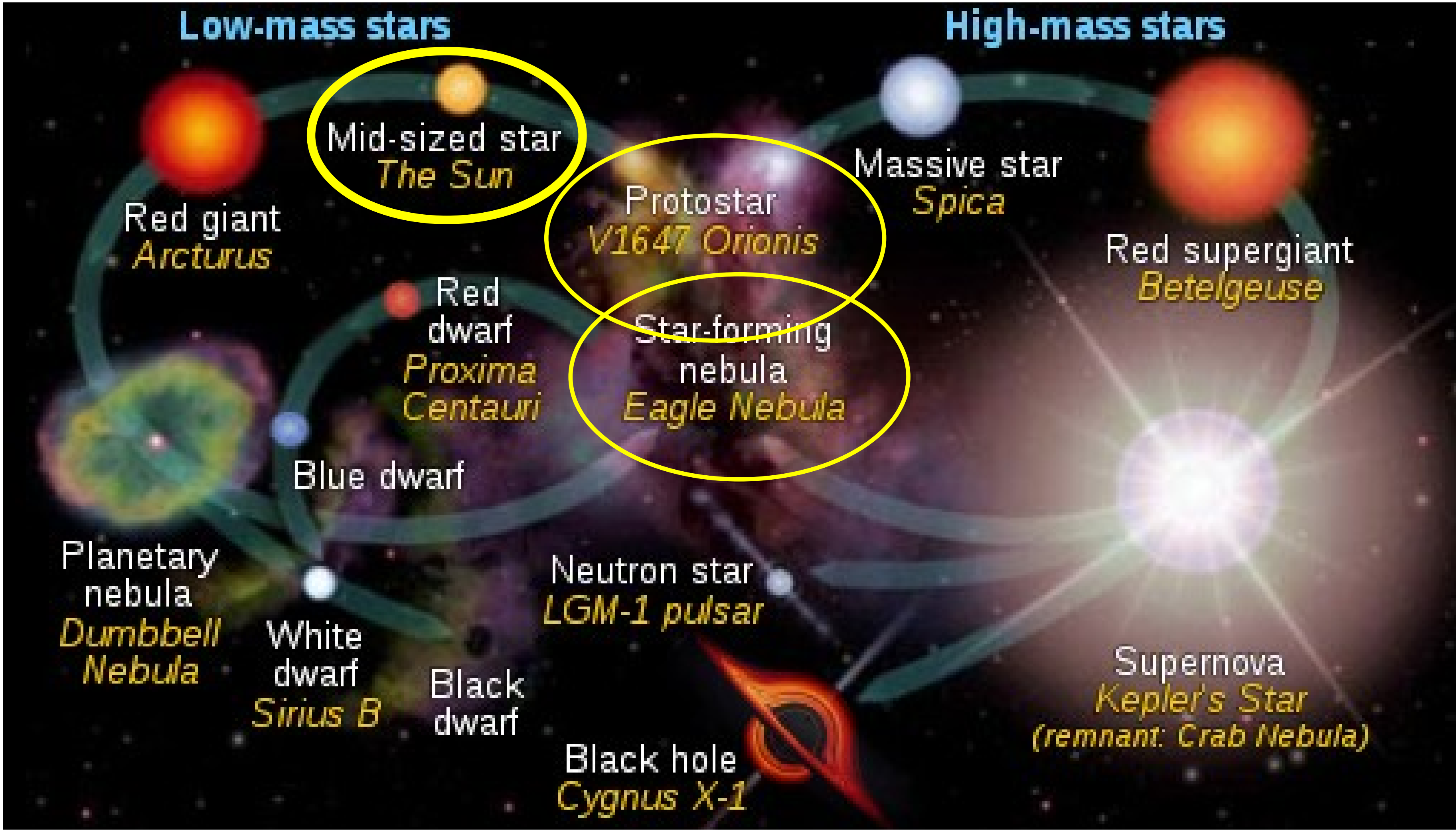
McNeil's nebula illuminated by protostar V1647 Orionis
optical image
d = 1500 light-years away
$R/R_{\odot} \sim 3$ & $M/M_{\odot} \sim 0.8$ & $L/L_{\odot} \sim 9$
→ density is insufficient for the fusion reactions
of hydrogen in helium to begin

V1647 Orionis is highly variable in luminosity. The nebula was not seen in 2005 -- 2008 years and in 2018 -- ...

Stellar evolution

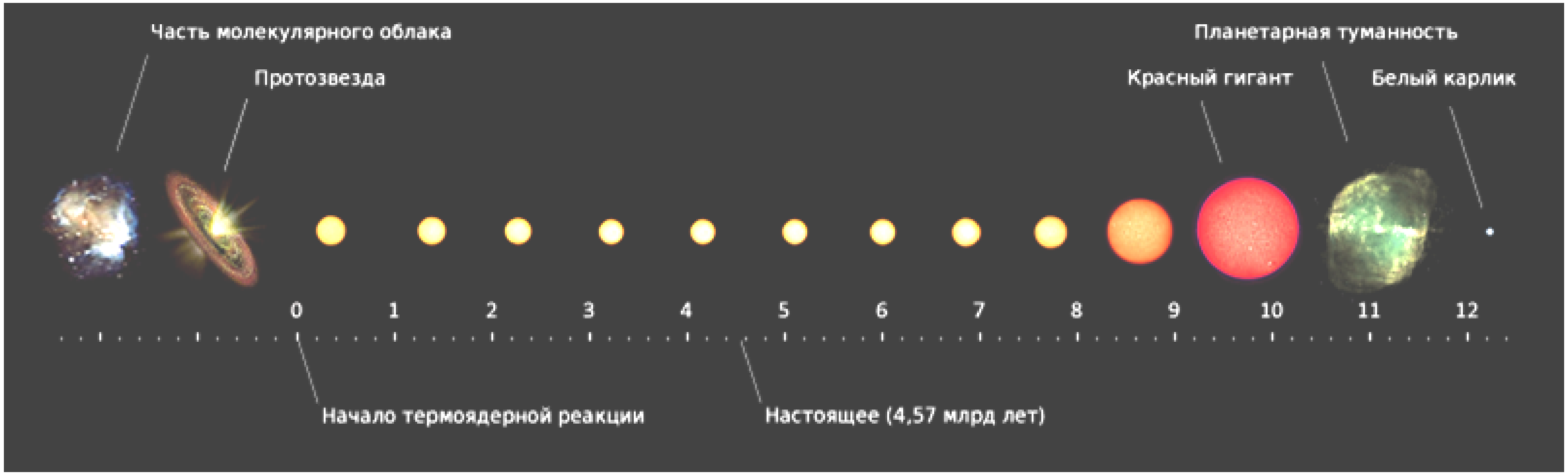


Stellar evolution



Stellar evolution

Past, present and future of the Sun (and Earth):



● **The good news is that the Sun will not explode as a supernova, because its mass is too low.**

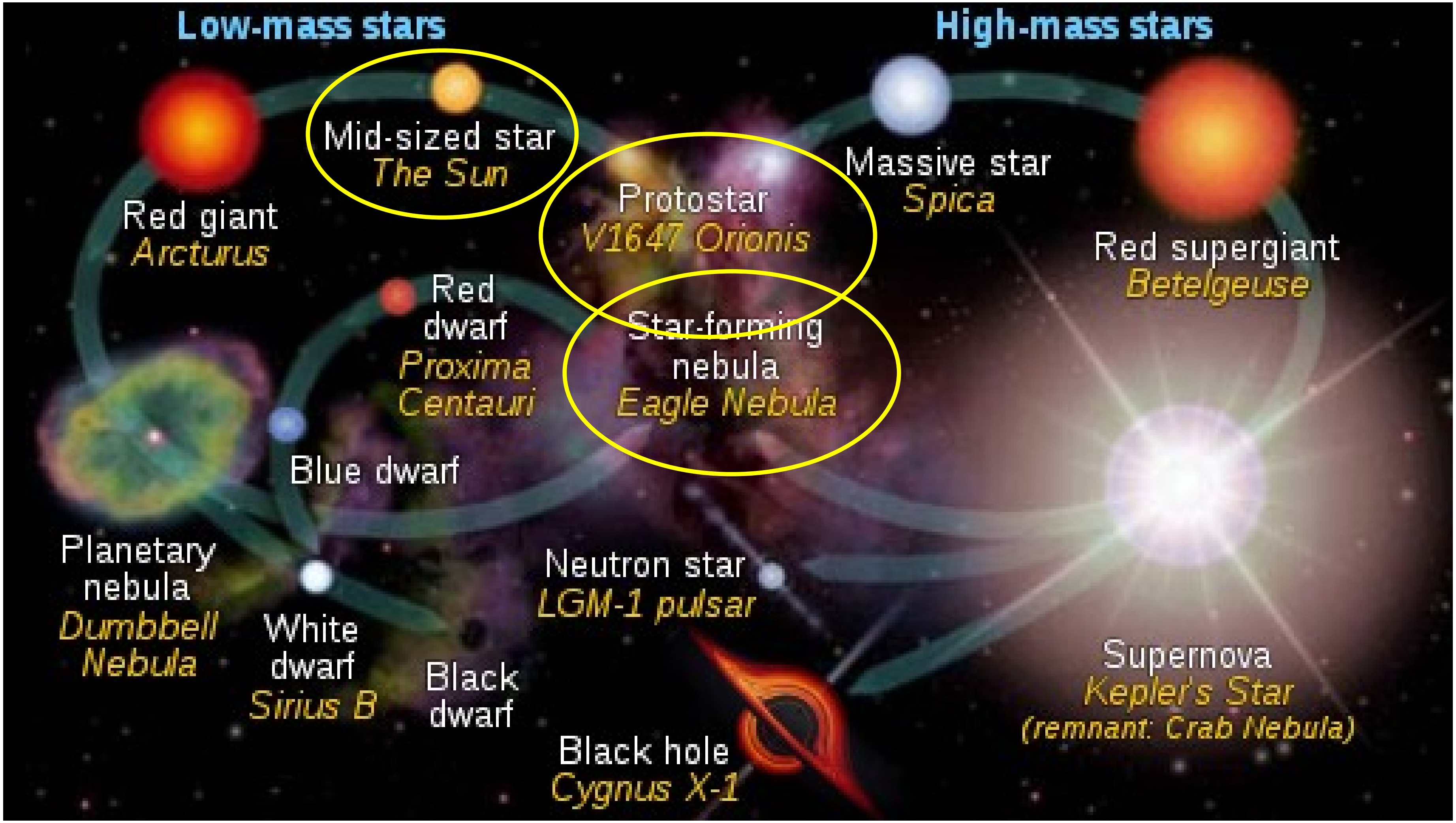
● In ~ 5 billion years, **Earth** will be inside the red giant, which **the Sun** will turn into.

$$R^{\text{read giant}}/R_{\odot} \sim 10^2 - 10^3 \text{ vs. } R_{\oplus}^{\text{orbit}}/R_{\odot} = 215$$

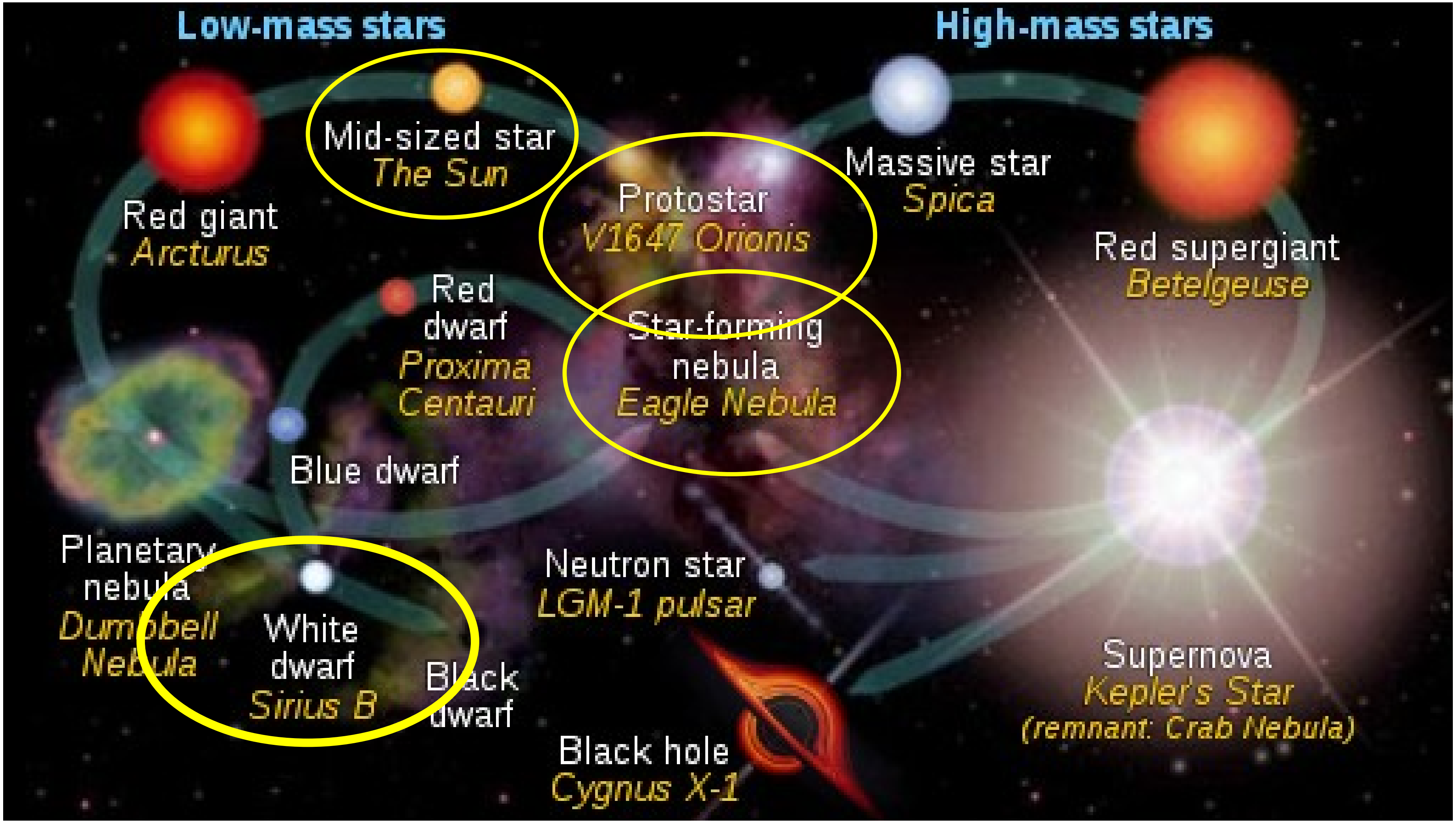
● In ~ 4.5 billion years, the **Andromeda – Milky Way** galactic collision is predicted to occur. **The good news is that the probability of influence of the collision to the evolution of our solar system is small.**

Anyway --- develop science in order to have an opportunity to move to another planetary system and/or galaxy

Stellar evolution

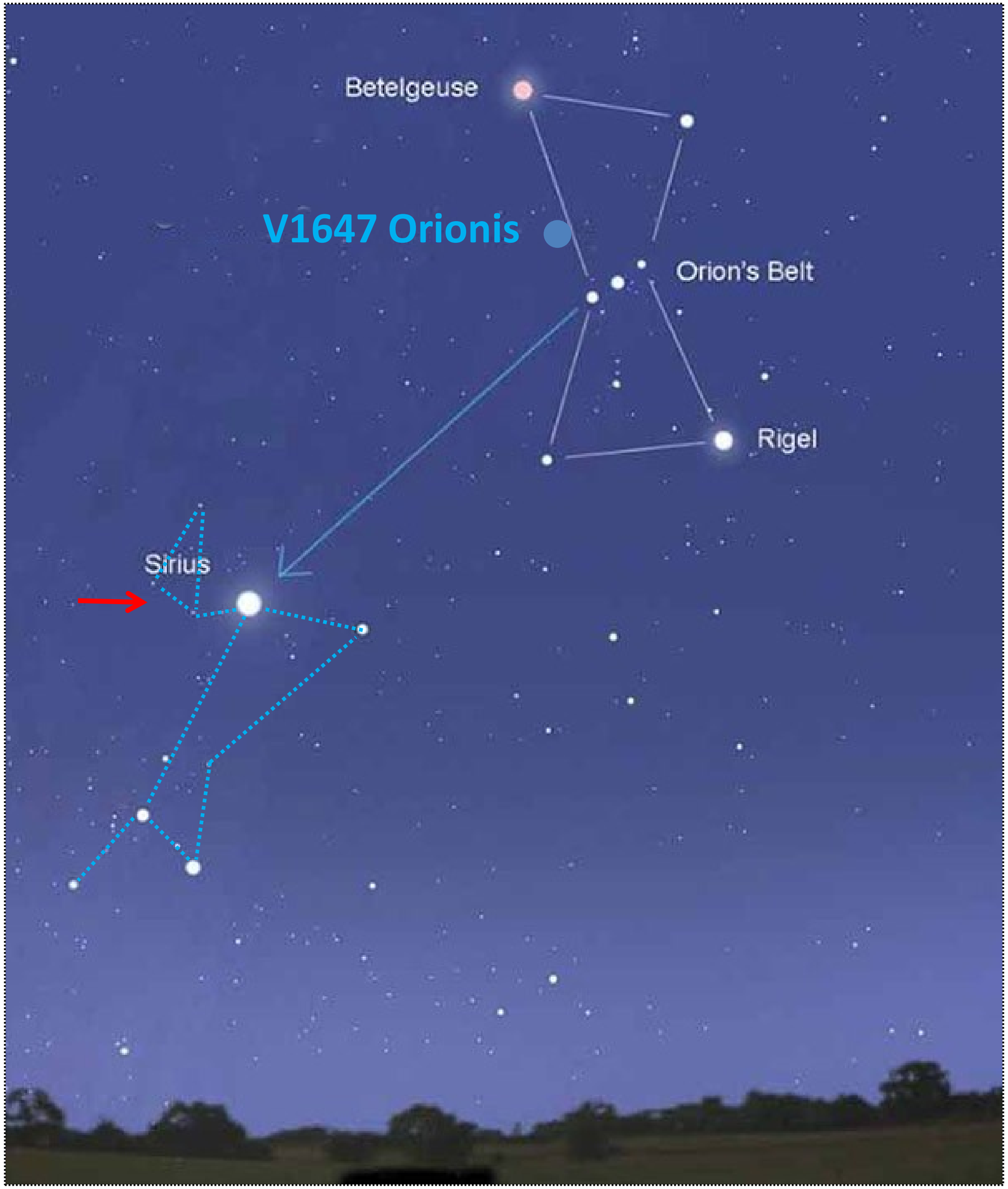


Stellar evolution



Canis (Dog) Major constellation & Orion constellation

“Star of stars”
(*Homer, the Iliad*)
“Morning star”
d = 8.6 light-years



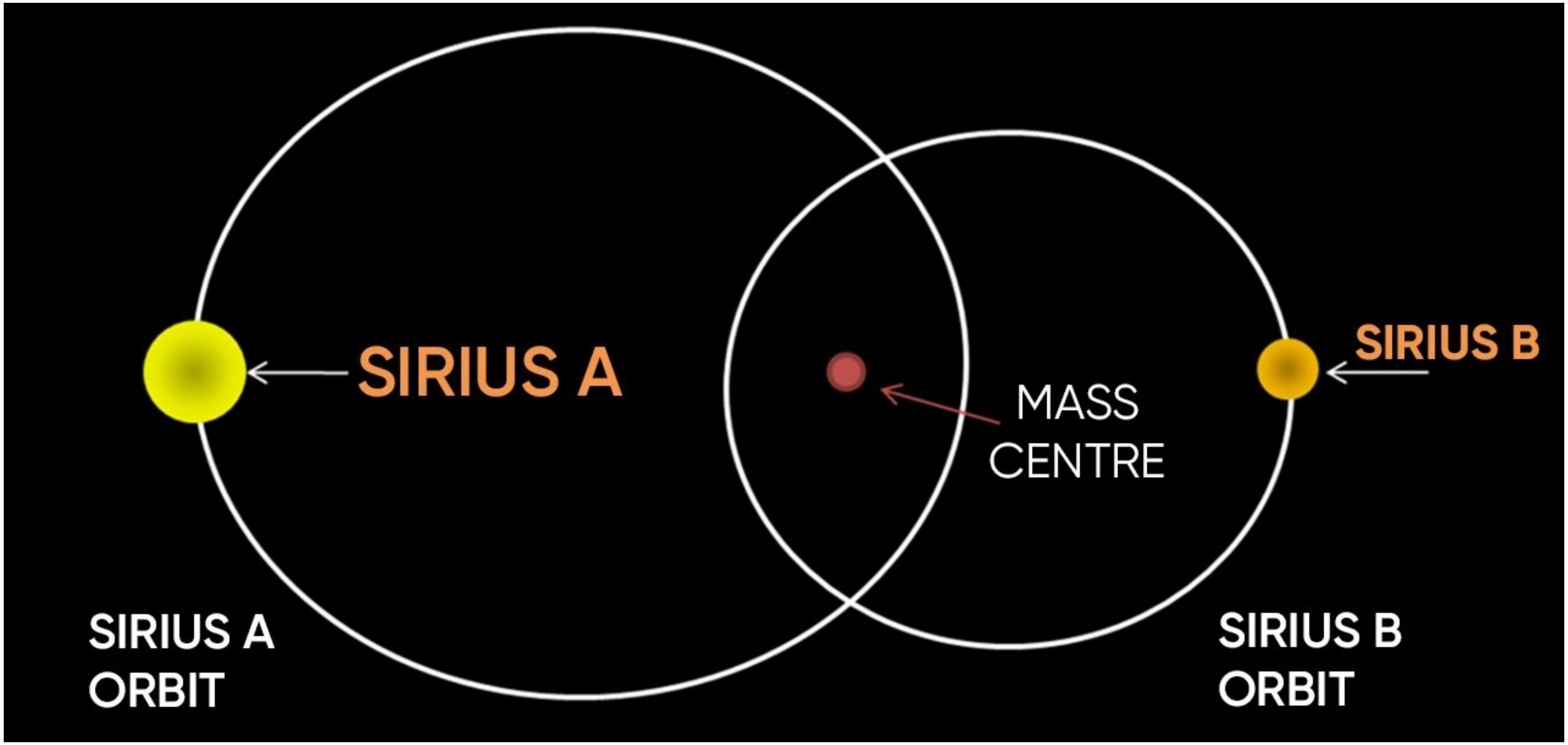
F. W. Bessel (1844) argued that Sirius is a binary system

Sirius A, Main sequence star: $M \sim 2M_\odot$ $R \sim 1.7R_\odot$ $L \sim 25L_\odot$

Sirius B, White dwarf: $M \sim \ M_\odot$ $R \sim R_\oplus$

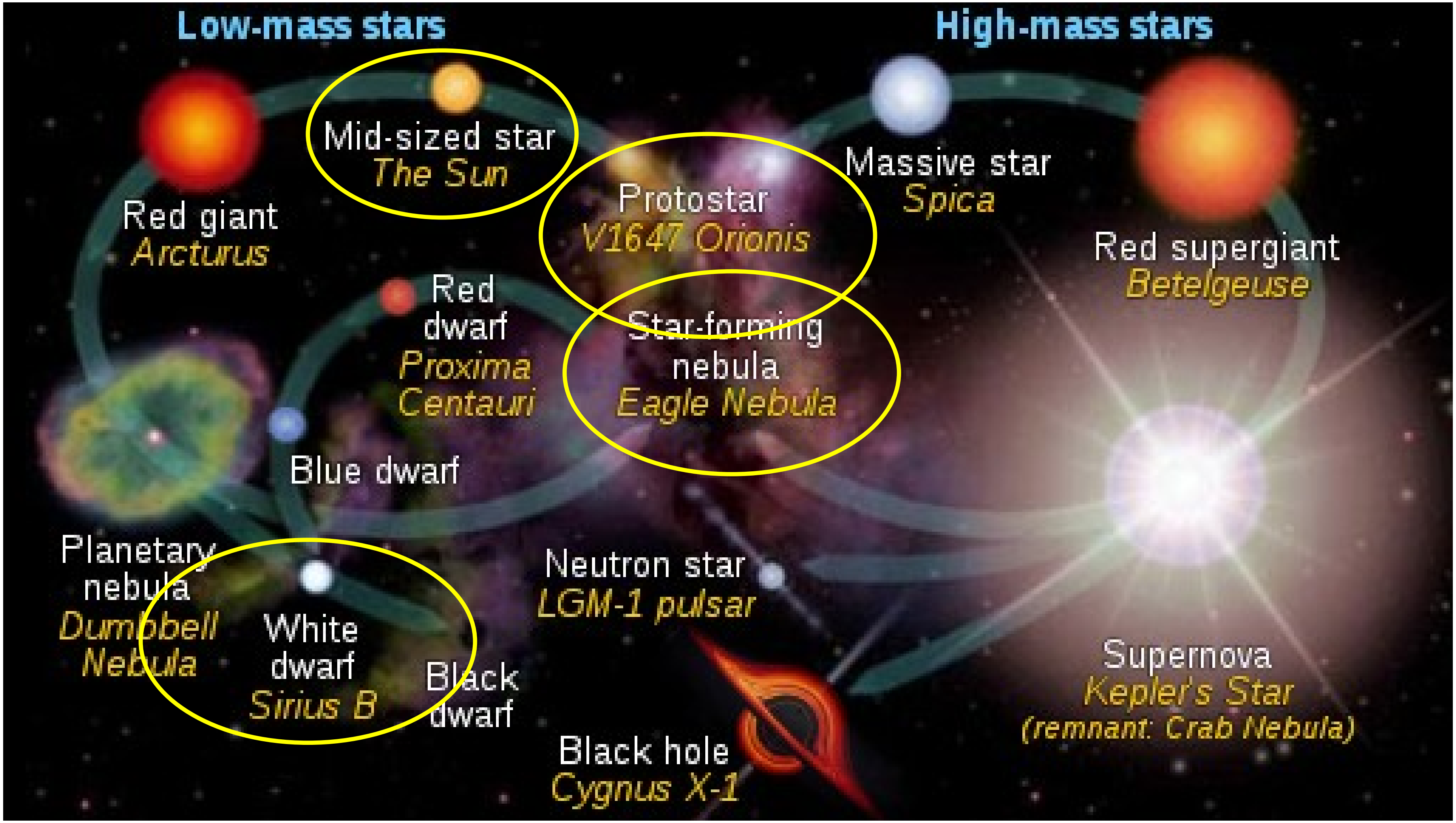
orbital period ~ 50 years, with a separation of about 20 AU

20 AU ~ the distance between Uranus and the Sun

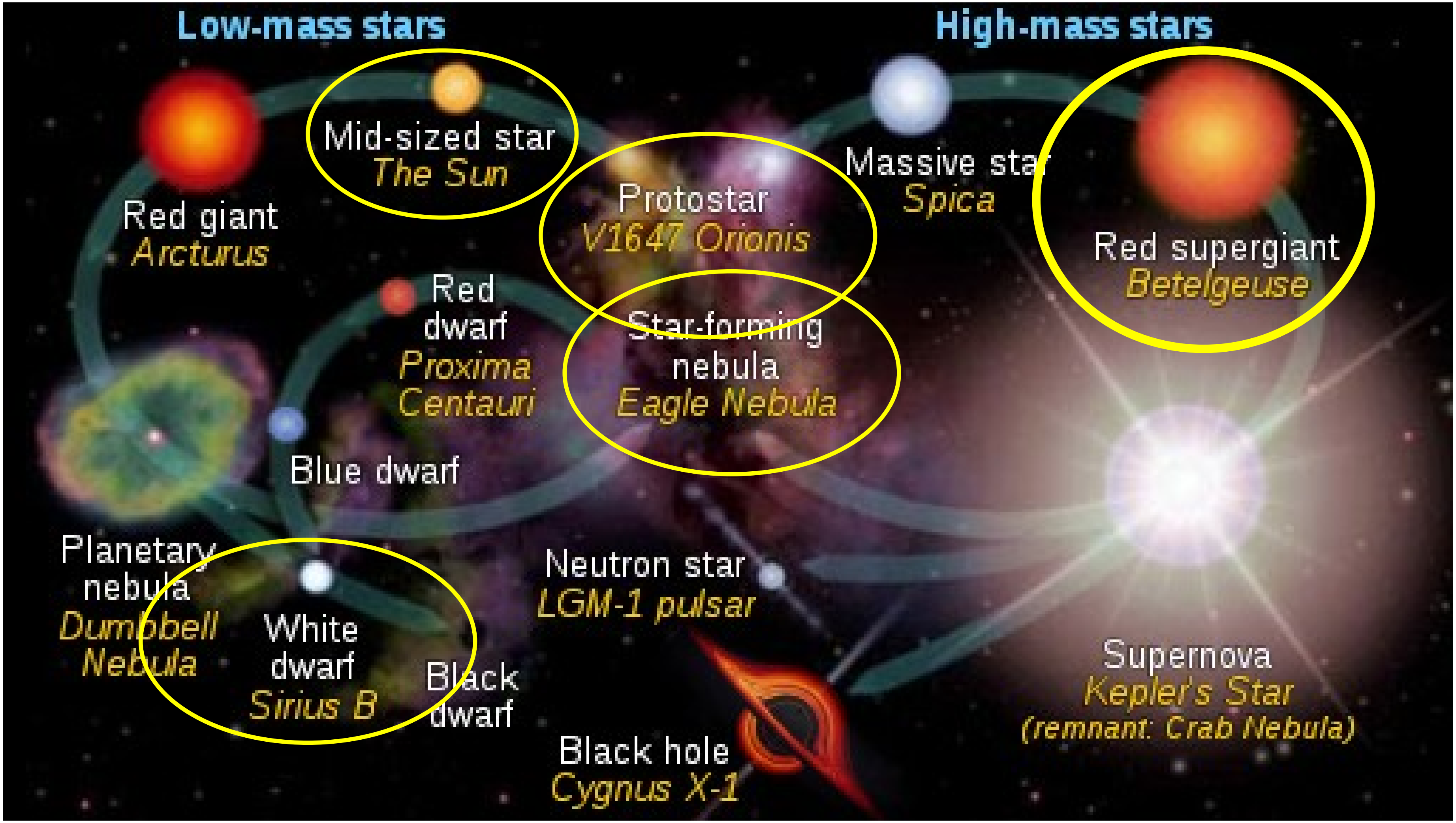


d = 8.7 light-years from the Earth

Stellar evolution

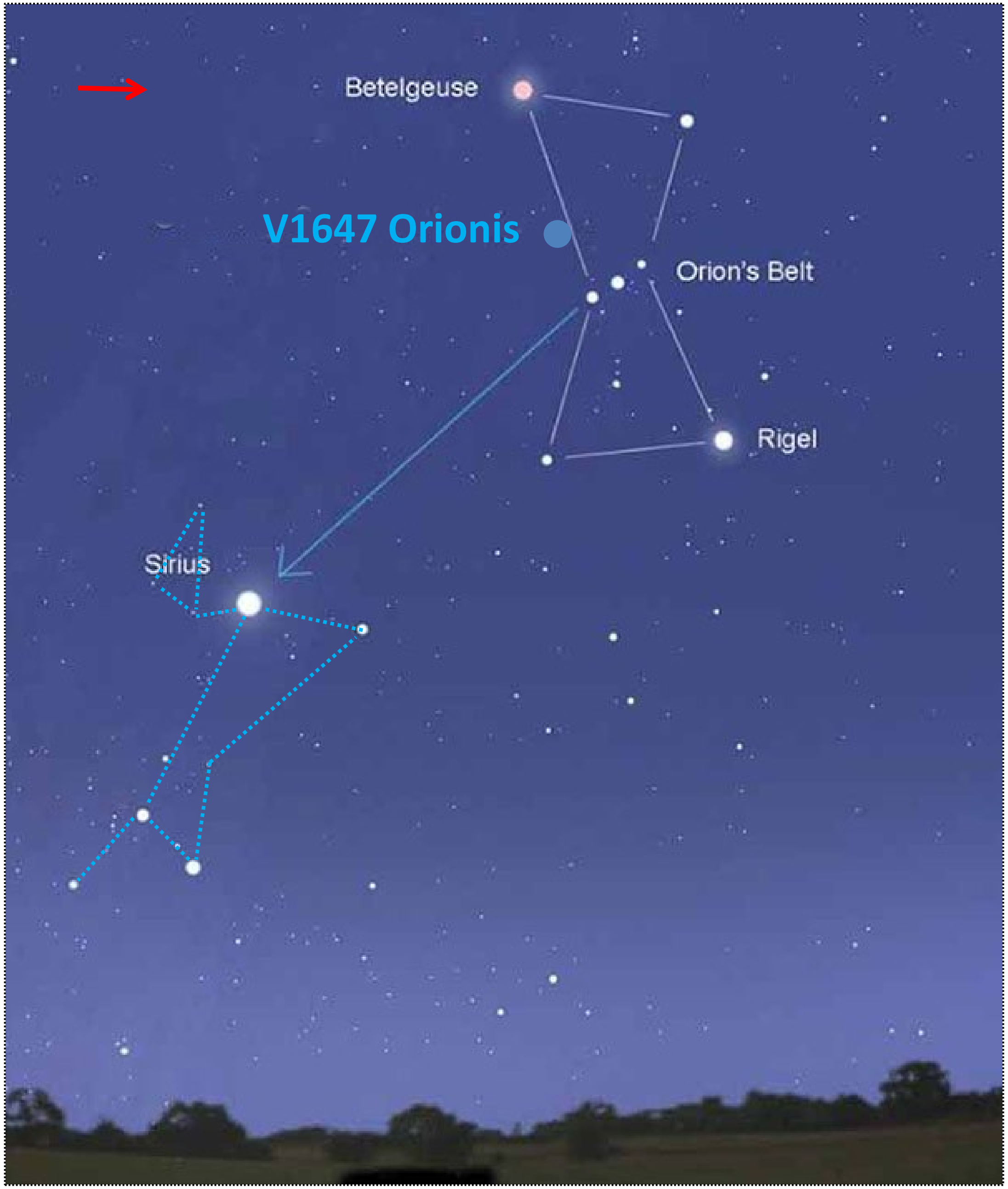


Stellar evolution



Canis (Dog) Major constellation & Orion constellation

Betelgeuse



Betelgeuse vs. The Sun

Name	Betelgeuse	The Sun
Galaxy	Milky Way	Milky Way
Star Type	Red supergiant star	Main Sequence Star
Mass ($M_\odot$)	~ 5	1
Age	~ 8 Myr	4.5 Gyr
Effective Temperature	~ 4000 K = 1 eV	5772 K
Colour	Red	Yellow (Atmosphere) White (in Space)
Constellation	Orion	N/A
Naked Eye Visible	Yes	Yes (but don't look at it!)
Radius ($R_\odot$)	~ 240	1
Luminosity ($L_\odot$)	~ 37000	1
Distance from Earth	~ 500 Light Years	500 Light Seconds

NB: $R_\oplus^{orbit}/R_\odot = 215$

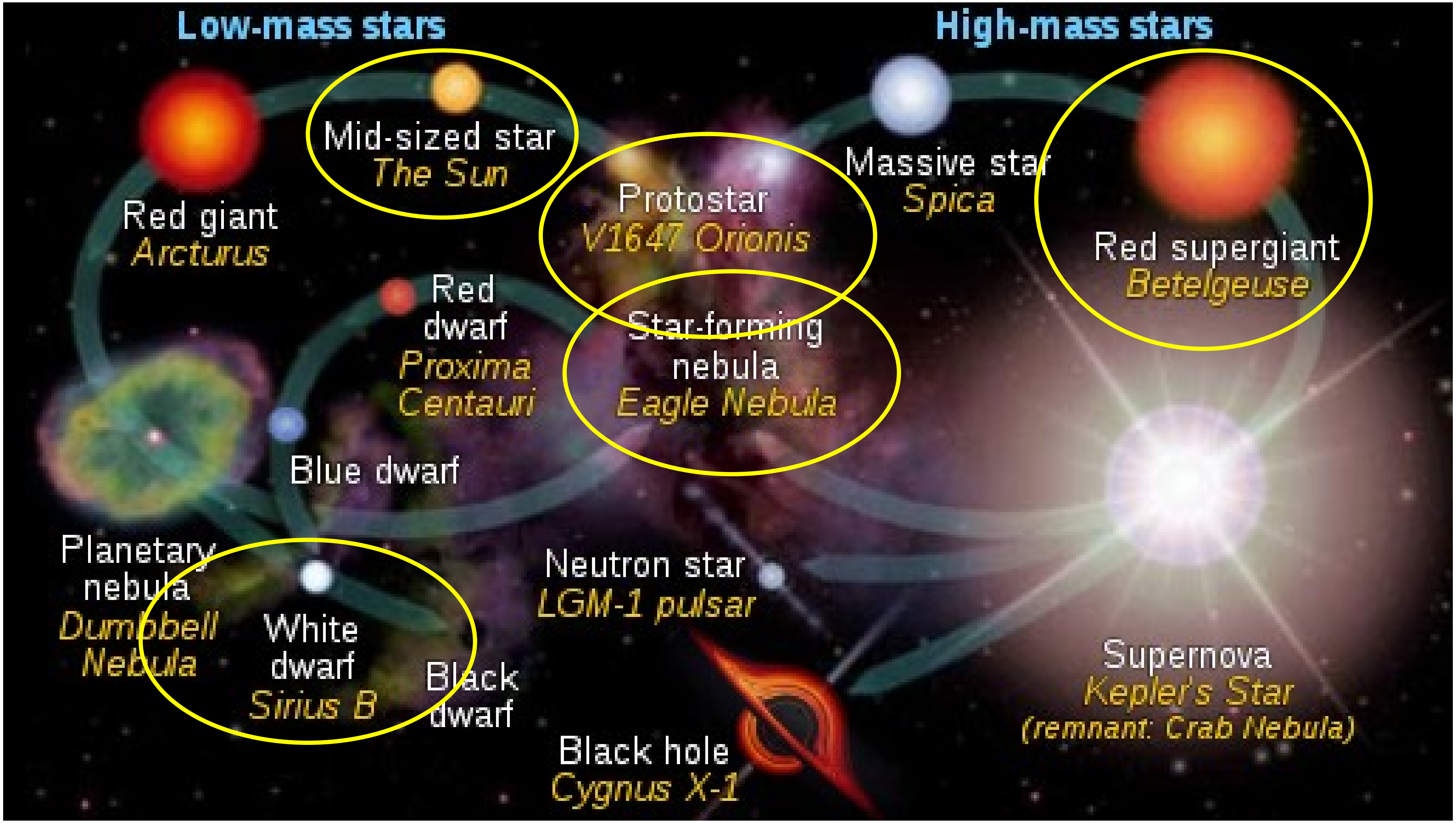
NB: Distance from Earth, # neutrinos = $(d_{SN}/d_{Betelgeuse})^2$

Betelgeuse vs. SN1987A: 550 : 168000 → 10^6 more detected neutrinos !!

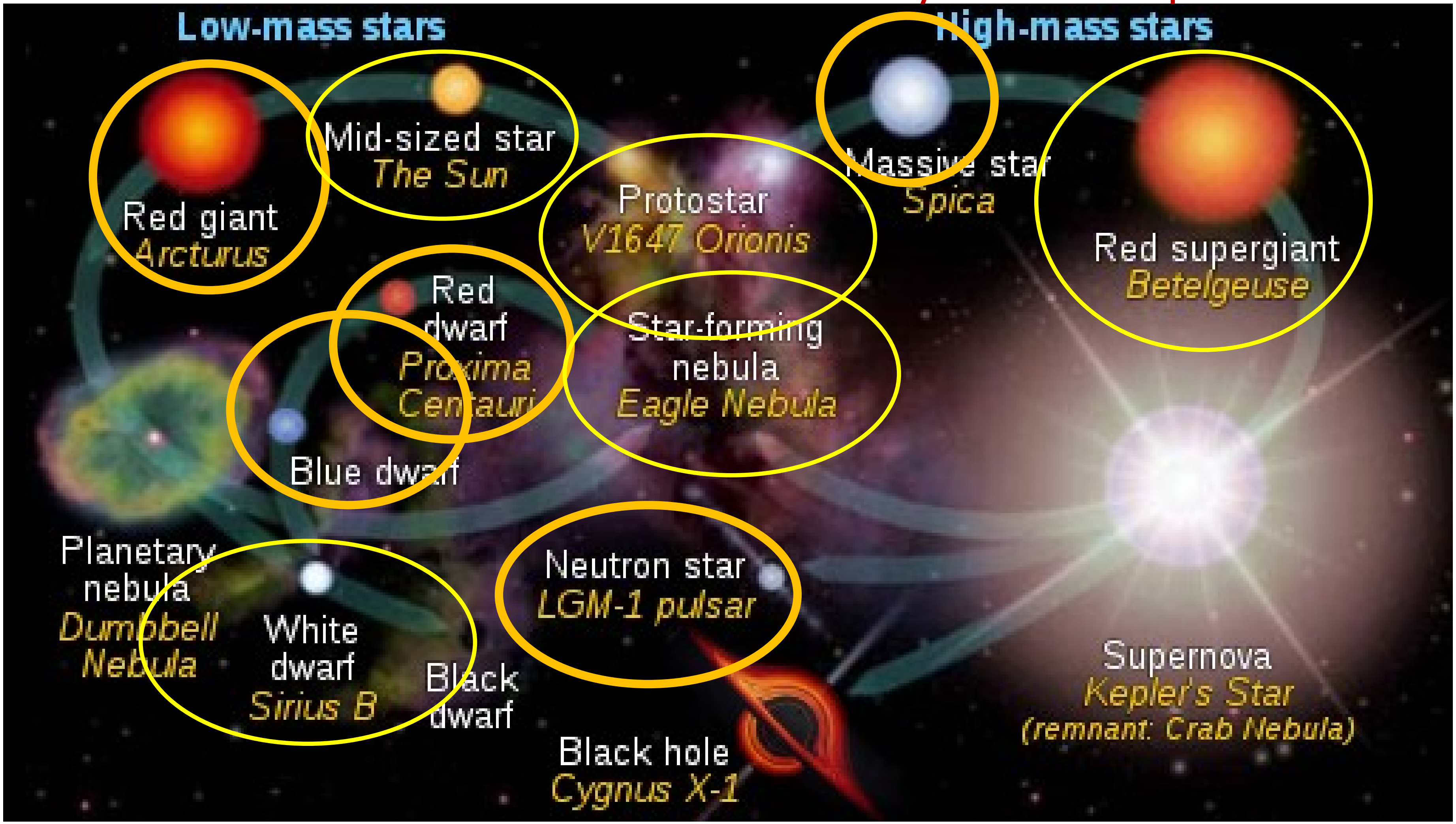
Kamiokande II (1987) → SuperKamiokande 3000 → 50000 tons of pure water

10 events → 10^8 events

Stellar evolution



Stellar evolution: Theory described applies to all objects in hydrostatic equilibrium



ELECRIC CHARGE IN STARS

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Lectures plan:

1. **The formation, life and death of stars.**
2. **Rosseland's paper (1924) with a particular solution of hydrostatic equilibrium for a self gravitating two-component charged fluid.**
3. **Singular differential equations. Poincare's theorem on analyticity. Dyson's argument.**
4. **General solution.**
5. **Relativistic theory of self gravitating multicomponent charged fluid.**
6. **Maximum electric charge of galaxy.**
7. **Open questions.**

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2/2

WHAT ARE STARS?

EQUILIBRIUM EQUATIONS

MODEL SOLUTIONS

FROM THE FIRST LECTURE, IT SHOULD BE TAKEN OUT THAT:

1) The origin of stars begins in giant molecular clouds.

2) The clouds are fragmented into smaller parts due to the Jeans instability.

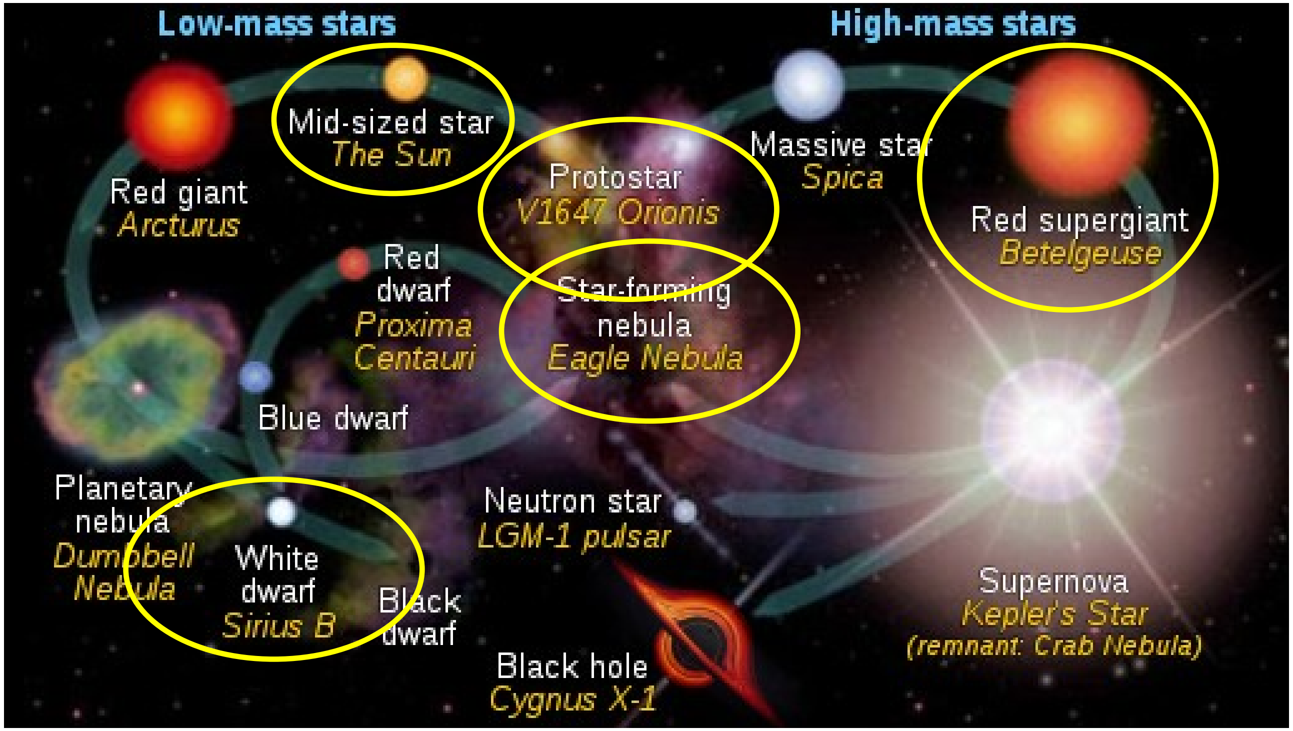
$$(*) \quad E_{\mathrm{kinetic}} \sim \frac{M}{m_{\mathrm{H}}} \qquad \sim \frac{GM^2}{R}$$

$$(**) \quad M \sim$$

$$\boxed{\Rightarrow M \geq M_{\mathrm{J}} \sim \left(\frac{kT}{Gm_{\mathrm{H}}}\right)^{3/2} \rho^{-1/2}}$$

3) The fragmentation process ends with formation of a protostar when the substance becomes opaque, with a mass 0.01 $M_{\odot}$ < M < 30 $M_{\odot}$.

4) Further evolution depends on the mass.



a) The left track, to which the Sun belongs, describes the evolution of stars with $M < 8\ M_{\odot}$. These stars go through the stages of the main sequence, a planetary nebula, a red giant and a white dwarf.

b) The right track describes the evolution of more massive stars. At the final stage after the formation of a red supergiant, it ends its life with a supernova explosion, after which a neutron star, or black hole, is formed.

We have considered several examples of stars at various stages of their evolution.

We specified location of the object in our Galaxy, location in the sky, view in optic and the main parameters: mass, distance from the Sun etc.

Non-relativistic theory with Newton gravity

Ernest Rutherford & W.D. Harkin, chemist 1920: conjecture on neutrons in nuclei
James Chadwick 1932: detection of free neutrons
Arthur Eddington 1920: conjecture on
burning hydrogen to helium as a source of energy of stars

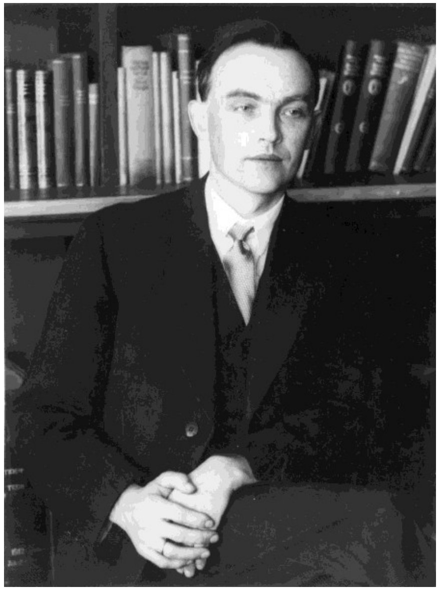
$\Delta E = \Delta m_{\text{nucl.}} c^2$

Gravitational compression vs. Nuclear fusion:

$$t \sim W_{gr}/L_\odot \sim (GM_\odot^2/R_\odot)/L_\odot \sim 3\times 10^7 \text{ years}$$

$$t \sim \Delta m_{\text{nucl.}} c^2 M_\odot/(m_p L_\odot) \sim 10^{10} \text{ years}$$

Arthur Eddington



Svein Rosseland

Mr. S. Rosseland,

Electrical State of a Star. By S. Rosseland.

(*Communicated by Prof. A. S. Eddington.*)

M.N., 84, 308, 1924.

Non-relativistic theory with Newton gravity

Two-component charged fluid:

$$\begin{cases} N_e = N_e^{\circ} \exp \dfrac{(-m\phi + e\psi)}{kT} \\ N_p = N_p^{\circ} \exp \dfrac{(-M\phi - Ze\psi)}{kT} \\ \rho_e = -\dfrac{1}{4\pi} \nabla^2 \psi \\ \rho = +\dfrac{1}{4\pi G} \nabla^2 \phi \end{cases}$$

$$\rho_e = eZN_p - eN_e$$

$$\rho = MN_p + mN_e$$

Non-relativistic theory with Newton gravity

Particular solution for:

$$\Rightarrow \frac{\psi}{\phi} = -\frac{M-m}{(Z+1)e}$$

$$\boxed{\frac{\rho_e}{e} \approx \frac{A^2}{Z\lambda_G}\frac{\rho}{\mathrm{M}} \qquad \lambda_G = \frac{e^2}{Gm_u^2} \approx 1.25 \times 10^{36}}$$

$$\rho_e = eZ\mathrm{N}_p - e\mathrm{N}_e$$

$$\rho = \mathrm{M}\mathrm{N}_p + m\mathrm{N}_e$$

Non-relativistic theory with Newton gravity

Questions appear:

1. **Poisson equations are second-order equations.**
   The second solution has been lost.

2. **Status of a locally neutral substance from the point of view of differential equations?**

Relativistic theory with GRT

PHYSICAL REVIEW D VOLUME 12, NUMBER 10 15 NOVEMBER 1975

Internal structure of multicomponent static spherical gravitating fluids

E. Olson and M. Bailyn
Physics Department, Northwestern University, Evanston, Illinois 60201
(Received 26 March 1975)

One-component neutral fluid

unknown functions	#	equations		#
n number density of particles	(1)	$u^\mu u_\mu = 1$	fixing proper time	(1)
u^μ four velocity of fluid	(4)	$C(g^{\mu\nu})^\sigma = 0$	coordinate conditions	(4)
$g^{\mu\nu}$ metric tensor	(10)	$Z^{\mu\nu} = 0$	Hilbert-Einstein eqs.	(10)
	15			15

Relativistic theory with GRT

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	15			15

Two-component neutral fluid

unknown functions	#
n_1 number density of particles	(1)
n_2 number density of particles	(1)
u^μ four velocity of fluid	(4)
$g^{\mu\nu}$ metric tensor	(10)
	16 = 15 + 1

Prescription: Add to the 15 field equations

THE MINIMUM TOTAL ENERGY REQUIREMENT

$$\delta E/\delta n_i = 0 \text{ with } \delta N_i = \delta \int n_i dV = 0$$

with respect to variations of the partial densities n_i
The variation gives essentially Gibbs' conditions:

$$\mu_i + m_i \varphi_G = \text{constant} \quad \text{(non-relativistic form)}$$

N-component charged fluid in chemical equilibrium

field eqs.

Gibbs

TOV equation

$u^\mu u_\mu = 1$ fixing proper time (1)

$C(g^{\mu\nu})^\sigma = 0$ coordinate conditions (4)

$Z^{\mu\nu} = 0$ Hilbert-Einstein eqs. (10)

$F^{\mu\nu}{}_{;\nu} = J^\mu$ **Maxwell eqs.** (4)

$\delta E/\delta n_i = 0$ under global conditions $\delta N_i = 0$

& $\boxed{\text{local } \Sigma_i \nu_{ai}\mu_i = 0 \text{ for } a = 1,\dots,M}$ (N - M)

---

18 + N − M + 1

n_i number density of particles with charge e_i

under the local constraints $\boxed{\Sigma_i \nu_{ai}\mu_i = 0 \text{ for } a = 1,\dots,M}$ (N - M)

u^μ four velocity of fluid (4)

$g^{\mu\nu}$ metric tensor (10)

A^μ **vector potential** (4)

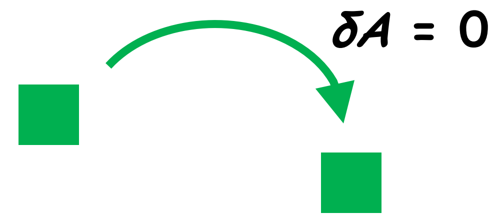
---

where $J^\mu = (\Sigma_i e_i n_i)u^\mu$ **18 + N - M**

EQUILIBRIUM EQUATIONS ARE THEREBY DETERMINATIVE

Gibbs' conditions (~ 1880) in Newton gravity, integral form;
gravity is a background

$$\mu_i + m_i \varphi_G = \text{constant}$$



O. Klein (1949) in GRT, integral form;
gravity is a background

$$(-g_{00})^{1/2}\mu_i = \text{constant}$$

Kodama and Yamada (1972) in GRT, differential form;
gravity depends on the matter

$\Sigma_i n_i$

$$\frac{d}{dr}\left(\frac{\partial\varepsilon}{\partial\rho_i}\right) = \frac{1}{2}\frac{\partial\varepsilon}{\partial\rho_i}\left\{\frac{d\lambda}{dr} - \kappa\sum_j \frac{\partial\varepsilon}{\partial\rho_j}\rho_j r e^{\lambda}\right\}.$$

→ TOV equation

Olson and Bailyn (1975) in GRT + EM, differential form;
gravity depends on the matter

$\Sigma_i n_i$

$$\left(\frac{\partial\rho_m}{\partial n_i}\right)' = \frac{q_i \mathcal{E} e^{\lambda/2}}{c^2 r^2} + \frac{\partial\rho_m}{\partial n_i}\left(\frac{\lambda'}{2} - \frac{4\pi G}{c^2} r e^{\lambda}\sum_j n_j \frac{\partial\rho_m}{\partial n_j}\right).$$

→ TOV + OB equation

ELECTROMAGNETIC FORCE
vs.
GRAVITATIONAL FORCE:

$$\lambda_{\rm G} = \frac{e^2}{G m_{\rm u}^2} \approx 1.25 \times 10^{36}$$

In complete screening → gravity plays the dominant role for stellar structure

Hydrostatic equilibrium equations are normally supplemented by:

LOCAL ELECTRO-NEUTRALITY CONSTRAINT

WHAT IS THE ORIGIN?

Hydrostatic equilibrium equations for a selfgravitating two-component charged fluid belong to the family of singular differential equations

THE MAIN BREAKTHROUGH OBSERVATION:

The origin of the local electro-neutrality is similar to:

$$A_\varepsilon : \varepsilon \frac{\mathrm{d}u}{\mathrm{d}x} = -u + x, \quad 0 \leqslant x \leqslant 1; \quad u(0) = 1.$$

$$A_0 : 0 = -u + x$$

= A constraint, leading to the loss of initial condition!

The example is from the monograph of MSU mathematicians:

Васильева Аделаида Борисовна
Бутузов Валентин Федорович

АСИМПТОТИЧЕСКИЕ МЕТОДЫ
В ТЕОРИИ СИНГУЛЯРНЫХ ВОЗМУЩЕНИЙ

How fundamental is LEC in physics?

- **ABSOLUTELY FUNDAMENTAL FOR ISOLATED SYSTEMS (thermodynamics)**
- **HOWEVER DOES NOT APPLY TO SYSTEMS IN EXTERNAL FIELDS incompatible with multi-component Gibbs' conditions**

HYDROSTATIC EQUILIBRIUM EQUATIONS FOR A SELFGRAVITATING TWO-COMPONENT CHARGED FLUID BELONG TO THE FAMILY OF SINGULAR DIFFERENTIAL EQUATIONS

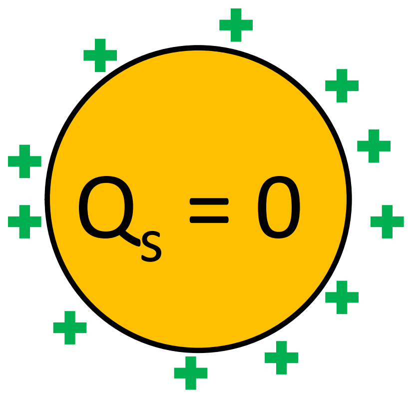
The problem is to construct the general solution

SOME OF THE PROPERTIES OF THE GENERAL SOLUTION CAN BE FORESEEN ON THE BASIS OF THE FOLLOWING SIMPLE ARGUMENTS:

(TWO HINTS ☺)

MAXIMUM CHARGE OF STAR

Ionosphere

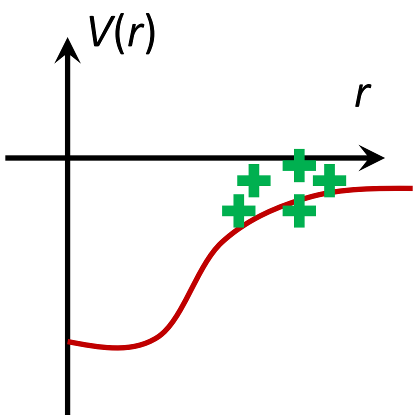


$$\frac{e^2 \Delta N_p^{\,2}}{R_s} <> \frac{G M_s \Delta N_p m_p}{R_s}$$

$$\Rightarrow \Delta N_p \le \frac{G M_s m_p}{e^2} \sim$$

$$\boxed{0 < Q_s < 150\,\mathrm{C}.}$$

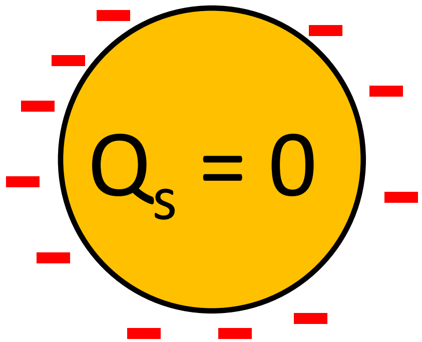
EXPECTED TO FOLLOW FROM THE GENERAL SOLUTION



See also: *How much electric surcharge fits on … a "white dwarf" star?*
American Journal of Physics 89, 291 (2021).
Parker Hund and Michael K.-H. Kiessling

MAXIMUM CHARGE OF STAR

Electrosphere

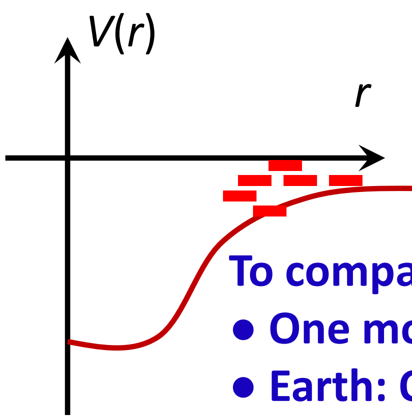


$$\frac{e^2 \Delta N_e^2}{R_s} \;<>\; \frac{G M_s \Delta N_e m_e}{R_s}$$

$$\Rightarrow \Delta N_e \leq \frac{G M_s m_e}{e^2} \sim \frac{M_s}{M_\odot}.$$

$$\boxed{-0.1\,\mathrm{C} < Q_s < 150\,\mathrm{C}.}$$

Expected to follow from the general solution

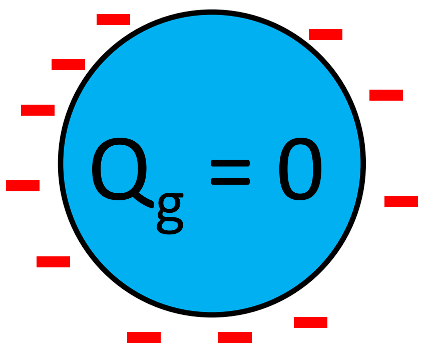


To compare:

- **One mole of ^{12}C nuclei has the total charge of Q = 600000 C.**
- **Earth: Q = - (40000 ... 57000) C >> 0.0003 C = 10^{21} e($M_\oplus/M_\odot$)**
- **The solar charge |Q| < (0.4 - 1)×10^{18} C, Iorio (2012).**

MAXIMUM CHARGE OF GALAXY

Electrosphere

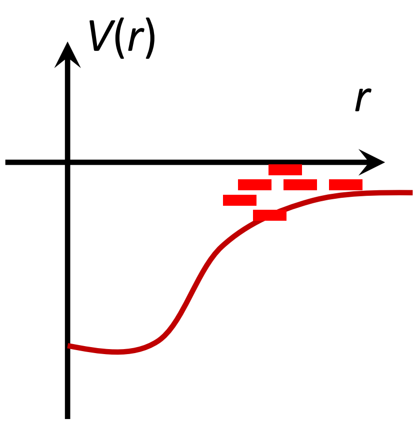


$$\frac{e^2 \Delta N_e^{\ 2}}{R_g} <> \frac{G M_g \Delta N_e m_e}{R_g}$$

$$\Rightarrow \Delta N_e \leq \frac{G M_g m_e}{e^2} \sim \frac{M_g}{M_{\text{Milky Way}}}.$$

$$\boxed{-10^{11}\ \text{C} < Q_g < 10^{14}\ \text{C}.}$$

Electromagnetic interaction of galaxy and star:



$W_{\text{potential}} \sim$

$\sim \quad \ll$

NB: $R_{\text{Milky Way}} \sim 10^5$ **light-years**

$M_{\text{Milky Way}} \sim 10^{12} M_\odot$

$T_\odot^{\text{core}} \sim 10^7$ **K**

Hydrostatic equilibrium equations for a selfgravitating two-component charged fluid belong to the family of singular differential equations

SECOND HINT:

The general solution is not regular in the gravitational constant G at $G=0$, as indicated by

- **the Poincare theorem on analyticity and**
- **Dyson's argument**

HELPFUL IN CONSTRUCTING THE GENERAL SOLUTION

DETAILS PROVIDED LATER

WHAT ARE STARS?

EQUILIBRIUM EQUATIONS

MODEL SOLUTIONS

WHAT ARE STARS?

EQUILIBRIUM EQUATIONS

MODEL SOLUTIONS

A. Two-fluid model

PHYSICAL REVIEW D **97**, 083016 (2018)

Hydrostatic equilibrium of stars without electroneutrality constraint

M. I. Krivoruchenko,[1,2] D. K. Nadyozhin,[1,3] and A. V. Yudin[1,*]
[1]*Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya 25, 117218 Moscow, Russia*
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[3]*Kurchatov Institute, Akademika Kurchatova Pl. 1, 123182 Moscow, Russia*

Starts from multi-component non-relativistic Gibbs' condition:

$$\boxed{\begin{aligned} \mu_{\rm i} + m_{\rm i}\varphi_{\rm G} + Ze\varphi_{\rm E} &= \text{const}, \\ \mu_{\rm e} + m_{\rm e}\varphi_{\rm G} - e\varphi_{\rm E} &= \text{const}, \end{aligned}}$$

i = ions, $m_i = Am_u$, m_u = 931 MeV
e = electrons , m_e = 0.51 MeV
φ_E = electro-static potential
φ_G = gravitational potential

Also discussed in:
Electrically nonneutral ground states of stars
PHYSICAL REVIEW D 103, 043004 (2021)
Parker Hund and Michael K.-H. Kiessling

A. Two-fluid model

Rosseland equations T > 0

$$\left\{\begin{aligned} N_e &= N_e^\circ \exp\frac{(-m\phi + e\psi)}{kT} \\ N_p &= N_p^\circ \exp\frac{(-M\phi - Ze\psi)}{kT} \\ \rho_e &= -\frac{1}{4\pi}\nabla^2\psi \\ \rho &= +\frac{1}{4\pi G}\nabla^2\phi \end{aligned}\right.$$

Two-fluid model T ≥ 0

$$\left\{\begin{aligned} \mu_i + m_i\varphi_G + Ze\varphi_E &= \text{const}, \\ \mu_e + m_e\varphi_G - e\varphi_E &= \text{const}, \\ \Delta\varphi_G &= 4\pi G\rho_m, \\ \Delta\varphi_E &= -4\pi\rho_e, \end{aligned}\right.$$

$$\begin{aligned} \rho_m &= m_i n_i + m_e n_e, \\ \rho_e &= Zen_i - en_e. \end{aligned}$$

$$\boxed{\begin{aligned} \Delta_r(Z\mu_e + \mu_i) &= -4\pi G(m_i + Zm_e)\rho_m, \\ \Delta_r(m_i\mu_e - m_e\mu_i) &= -4\pi e(m_i + Zm_e)\rho_e. \end{aligned}}$$

A. Two-fluid model

Equaiton of state of matter with polytropes:

- **Pressure** $$P_{\rm k} = K_{\rm k} n_{\rm k}^{1+1/\eta_{\rm k}}, \quad {\rm k} = ({\rm i}, {\rm e})$$
- **Density** $$n_{\rm k} \equiv n_{\rm k0} \theta_{\rm k}^{\eta_{\rm k}}$$
- **Chemical potential** $$\mu_k = \mu_{k0} \theta_k,$$ $$\mu_{k0} = K_k (1 + \eta_k) n_{k0}^{1/\eta_k}.$$
- **Dimensionless coordinate x:**

$$r = r_0 x, \quad r_0^2 = \frac{Z \mu_{\rm e0}}{4\pi G m_{\rm i}(m_{\rm i} + Z m_{\rm e}) n_{\rm i0}}$$

A. Two-fluid model

THE BASIC SYSTEM OF EQUATIONS

$$\boxed{\begin{aligned} \triangle_x(\theta_\mathrm{e} + \Lambda_\mathrm{i}\theta_\mathrm{i}) &= -(\theta_\mathrm{i}^{\eta_\mathrm{i}} + \Lambda_\mathrm{m}\Lambda_\mathrm{e}\theta_\mathrm{e}^{\eta_\mathrm{e}}) \\ \triangle_x(\theta_\mathrm{e} - \Lambda_\mathrm{m}\Lambda_\mathrm{i}\theta_\mathrm{i}) &= -\Lambda_\mathrm{G}(\theta_\mathrm{i}^{\eta_\mathrm{i}} - \Lambda_\mathrm{e}\theta_\mathrm{e}^{\eta_\mathrm{e}}) \end{aligned}}$$

с начальными условиями

$$\theta_\mathrm{k}(0) = 1, \qquad \theta_\mathrm{k}'(0) = 0.$$

и параметрами

$$\boxed{\Lambda_\mathrm{e} = \frac{n_\mathrm{e0}}{Z n_\mathrm{i0}} \approx 1 \quad \text{(= LEC)}}$$

$$\Lambda_\mathrm{m} = \frac{Z m_\mathrm{e}}{m_\mathrm{i}}$$

$$\Lambda_\mathrm{i} = \frac{\mu_\mathrm{i0}}{Z\mu_\mathrm{e0}} \approx \frac{(1+\eta_\mathrm{i})P_\mathrm{i0}}{(1+\eta_\mathrm{e})P_\mathrm{e0}} \ll 1$$

$$\boxed{\Lambda_\mathrm{G} = \frac{Z^2 e^2}{G m_\mathrm{i}^2} = \left(\frac{Z}{A}\right)^2 \lambda_\mathrm{G}.}$$

HYDROSTATIC EQUILIBRIUM EQUATIONS FOR A SELFGRAVITATING TWO-COMPONENT CHARGED FLUID BELONG TO THE FAMILY OF SINGULAR DIFFERENTIAL EQUATIONS

THE BASIC SYSTEM OF EQUATIONS

$$
\begin{aligned}
\triangle_x(\theta_e + \Lambda_i\theta_i) &= -(\theta_i^{\eta_i} + \Lambda_m\Lambda_e\theta_e^{\eta_e}) \\
\triangle_x(\theta_e - \Lambda_m\Lambda_i\theta_i) &= -\Lambda_G(\theta_i^{\eta_i} - \Lambda_e\theta_e^{\eta_e})
\end{aligned}
$$

$$\theta_k(0) = 1, \qquad \theta_k'(0) = 0.$$

$$0 \le x \le x_b.$$

1. **Cauchy problem for the 4-th order ODE**
2. **Singularly perturbed system, with $\varepsilon = 1/\Lambda_G <<< 1$.**
3. **Similar to the Tikhonov system of equations:**

$$
A_\varepsilon \left\{
\begin{aligned}
&\varepsilon \frac{dz}{dx} = F(z, y, x, \varepsilon), \quad \frac{dy}{dx} = f(z, y, x, \varepsilon), \\
&0 \leqslant x \leqslant x_0, \\
&z(0) = z^0, \quad y(0) = y^0.
\end{aligned}
\right.
$$

ROSSELAND-TYPE REGULAR SOLUTION TO THE UNCONSTRAINED HYDROSTATIC EQUILIBRIUM EQUATIONS

B. Two-fluid model with equal polytropic indices

R. Emden

A particular solution for $\theta_i(x) = \theta_e(x) = \theta(\tilde{x})$ and $\eta_i = \eta_e = \eta$ with

$$\tilde{x} = x\sqrt{\frac{1+\Lambda_\mathrm{m}\Lambda_\mathrm{e}}{1+\Lambda_\mathrm{i}}},$$

can be found from the Lane-Emden equation

$$\triangle_{\tilde{x}}\theta(\tilde{x}) = -\theta^\eta(\tilde{x})$$

Free (!) parameter Λ_e becomes function of Λ_G:

$$\Lambda_\mathrm{e} \;=\; \Lambda_\mathrm{e}^\mathrm{reg} = \frac{\Lambda_\mathrm{G}(1+\Lambda_\mathrm{i}) - (1-\Lambda_\mathrm{m}\Lambda_\mathrm{i})}{\Lambda_\mathrm{G}(1+\Lambda_\mathrm{i}) + \Lambda_\mathrm{m}(1-\Lambda_\mathrm{m}\Lambda_\mathrm{i})}.$$

NB: but Λ_e measures deviations from LEC

The limit $\Lambda_G \to \infty$ is smooth, the solution is regular.

Rosseland-type regular solution to the unconstrained hydrostatic equilibrium equations

The limit $\Lambda_G \to \infty$ is smooth, the solution is regular.

Similar to $A_\varepsilon : \varepsilon \dfrac{\mathrm{d}u}{\mathrm{d}x} = -u + x, \; 0 \leqslant x \leqslant 1; \quad u(0) = 1.$

$A_0 : 0 = -u + x$ **= A constraint leading to the loss of initial condition**

In the problem A_{reg} the regularity condition leads to $\Lambda_e = \Lambda_e^{reg}(\Lambda_G)$ and

the loss of the initial condition.

For $\Lambda_e = \Lambda_e^{reg}(\Lambda_G)$ the regular solution is exact.

$$x(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k (\underbrace{x_k(t)}_{\text{Taylor}} + \Pi_k(\tau)).$$

$\tau = t/\varepsilon,$

$\varepsilon = 1/\Lambda_G \ll$

0

Laurent: $f(\tau;\varepsilon) = \sum_{s=-\infty}^{+\infty} \varepsilon^s f_s(\tau)$

ROSSELAND-TYPE REGULAR SOLUTION TO THE UNCONSTRAINED HYDROSTATIC EQUILIBRIUM EQUATIONS

E. Global stellar parameters

The mass of the star takes the form

ρ_{m} = mass-energy density

$$M_{\mathrm{s}} = \int_0^{R_{\mathrm{s}}} 4\pi r^2 (n_{\mathrm{i}} m_{\mathrm{i}} + n_{\mathrm{e}} m_{\mathrm{e}}) dr = -4\pi r_0^3 n_{\mathrm{i}0} m_{\mathrm{i}} x^2 \frac{d}{dx} (\theta_{\mathrm{e}} + \Lambda_{\mathrm{i}} \theta_{\mathrm{i}}) \Big|_{x=x_{\mathrm{b}}}$$

$$r_0^2 = \frac{Z \mu_{\mathrm{e}0}}{4\pi G m_{\mathrm{i}} (m_{\mathrm{i}} + Z m_{\mathrm{e}}) n_{\mathrm{i}0}}$$

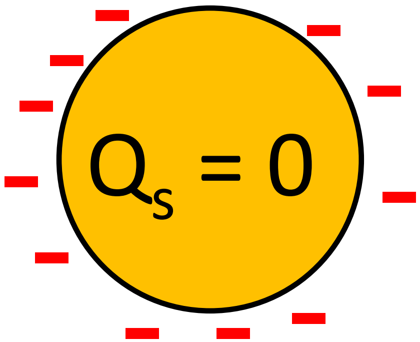
The total stellar charge (at the boundary of the component) is equal to

ρ_{e} = charge density

$$Q_{\mathrm{s}} = \int_0^{R_{\mathrm{s}}} 4\pi r^2 (Z e n_{\mathrm{i}} - e n_{\mathrm{e}}) dr = -4\pi r_0^3 n_{\mathrm{i}0} \frac{Ze}{\Lambda_{\mathrm{G}}} x^2 \frac{d}{dx} (\theta_{\mathrm{e}} - \Lambda_{\mathrm{m}} \Lambda_{\mathrm{i}} \theta_{\mathrm{i}}) \Big|_{x=x_{\mathrm{b}}}$$

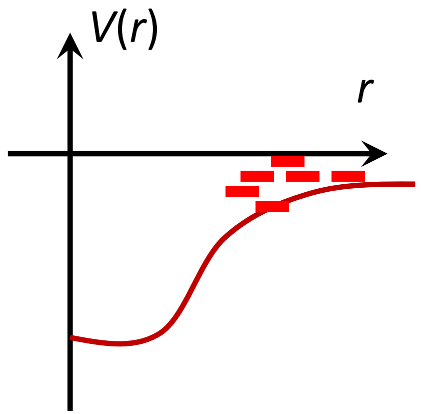
Electro- and ionosphere of stars in two-fluid model

Electron envelopes @ boundaries of

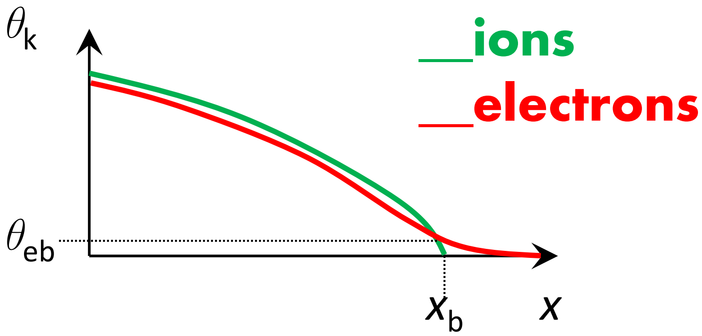


- solids → W ~ 3 eV
- strange stars → W ~ 30 MeV
- distinct phases

in nuclear matter → W ~ 30 MeV



How it looks in stars:



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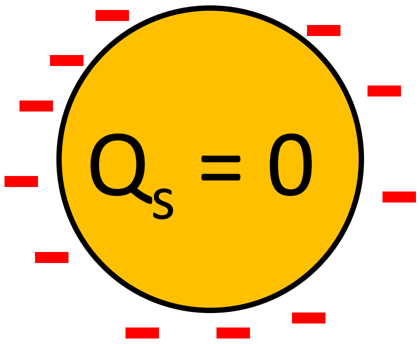

- solids → W ~ 3 eV
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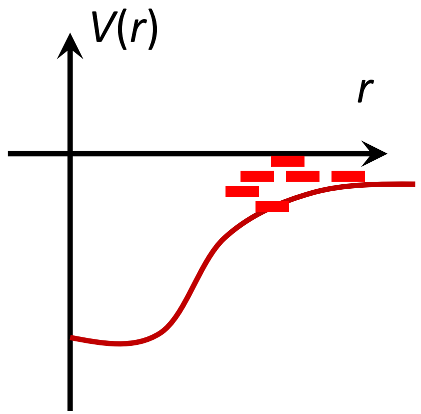

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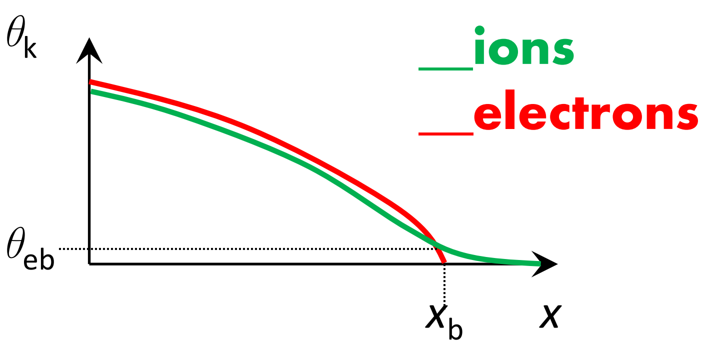


Electrosphere of stars in two-fluid model

A. A polytropic model

Start with Gibbs' condition:

$$\mu_{\rm e} + m_{\rm e}\varphi_{\rm G} - e\varphi_{\rm E} \;=\; {\rm const},$$

Applying Laplacian, we get the Thomas-Fermi equation:

$$\Delta_r \mu_{\rm e} = 4\pi e^2 n_{\rm e}\left(1 - \frac{G m_{\rm e}^2}{e^2}\right)$$

with the boundary conditions:

- μ_e is continuous
- μ_e' is continuous due to the balance of pressure & EM + gravity:

$$-\frac{1}{n_{\rm e}}\frac{dP_{\rm e}}{dr} = \frac{eQ_{\rm s}}{R_{\rm s}^2} + \frac{G m_{\rm e} M_{\rm s}}{R_{\rm s}^2}$$

Electrosphere of stars in two-fluid model

A. A polytropic model

Dimensionless units:

$$\lambda_{\rm m} = \frac{m_{\rm e}}{m_{\rm u}}$$

$$r = R_{\rm s} + r_{\rm a} y, \quad r_{\rm a}^2 = \frac{\mu_{\rm a0}}{4\pi e^2 \left(1-\lambda_{\rm m}^2/\lambda_{\rm G}\right) n_{\rm a0}}$$

The main equation:

$$\boxed{\frac{d^2\theta_{\rm a}}{dy^2} = \theta_{\rm a}^{\eta_{\rm e}}(y).}$$

with the boundary conditions $\theta_{\rm a}(0) = 1$

$$\left.\frac{d\theta_{\rm a}}{dy}\right|_{y=0} = -(q_{\rm s} + \lambda_{\rm m})\sqrt{\frac{P_{\rm un}}{(1+\eta_{\rm e})P_{\rm a0}(\lambda_{\rm G} - \lambda_{\rm m}^2)}}$$

where $P_{\rm un} = \frac{G M_{\rm s}^2}{4\pi R_{\rm s}^4}.$

THE PROBLEM IS THUS FORMULATED.

Electrosphere of stars in two-fluid model

A. A polytropic model

The main equation:

$$\boxed{\frac{d^2\theta_a}{dy^2} = \theta_a^{\eta_e}(y).}$$

looks like the 2nd Newton law, y is time,

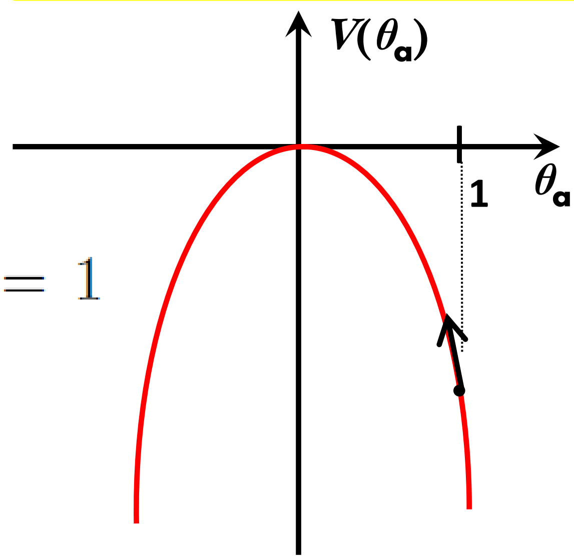
the potential and the energy:

$$\boxed{V(\theta_a) = -\frac{|\theta_a|^{1+\eta_e}}{1+\eta_e}}$$

$$\boxed{W = \frac{\theta_a'^2}{2} + V(\theta_a)}$$

$$\boxed{W_0 = \frac{\theta_a'^2(0)}{2} - \frac{1}{1+\eta_e}}$$

$$\theta_a(0) = 1$$



3 TYPES OF SOLUTIONS:

1. $W_0 > 0$ charged stars
2. $W_0 = 0$ neutral stars
3. $W_0 < 0$ *(compressed states)*

Electrosphere of stars in two-fluid model

A. A polytropic model

§70 *The Thomas–Fermi equation*

QUANTUM MECHANICS

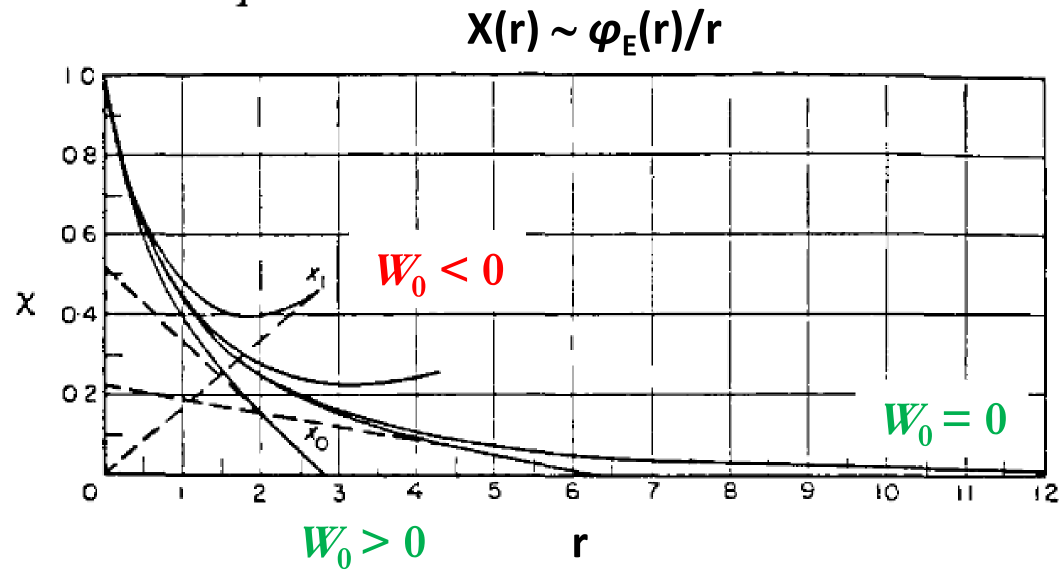
NON-RELATIVISTIC THEORY

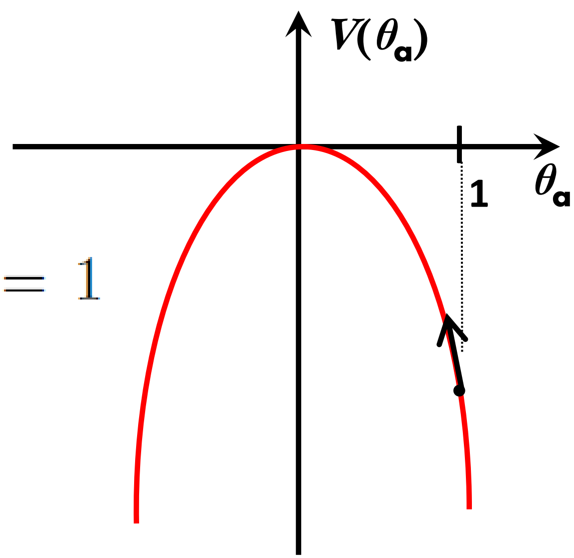
by

L. D. LANDAU AND E. M. LIFSHITZ

INSTITUTE OF PHYSICAL PROBLEMS, U.S.S.R. ACADEMY OF SCIENCES

Volume 3 of *Course of Theoretical Physics*





$$\theta_{\rm a}(0) = 1$$

3 TYPES OF SOLUTIONS:

1. $W_0 > 0$ charged stars
2. $W_0 = 0$ neutral stars
3. $W_0 < 0$ *(compressed states)*

Electrosphere of stars in two-fluid model

A. A polytropic model

Estimate of thickness of the electron envelope:

$\eta = 3/2$:
$$\frac{r_a}{R_s} \sim \left[\left(\frac{a_B}{R_s}\right)^3 \frac{m_u}{M_s}\lambda_G\right]^{1/5}$$
$$\approx 2.2\times 10^{-16}\left(\frac{R_\odot}{R_s}\right)^{3/5}\left(\frac{M_\odot}{M_s}\right)^{1/5}$$

$\boxed{y_a r_a \sim 10^6\ a_B}$ for $R_s = R_\odot$, $M_s = M_\odot$.

$\eta = 3$:
$$\frac{r_a}{R_s} \sim \sqrt{\frac{\lambda_G}{\alpha^{3/2}}\frac{m_u}{M_s}} \approx 1.3\times 10^{-9}\left(\frac{M_\odot}{M_s}\right)^{1/2}$$

$\boxed{y_a r_a \sim 1 \text{ millimeter}}$ for $R_s = 10$ km, $M_s = M_\odot$.

A. General properties of solutions

HINT #2

- **Poincare theorem on analyticity:**

Solutions to ODE systems, when they exist, are analytic functions of the initial coordinates and parameters in the region of analyticity of the ODEs.

Since analytic functions are determined by their singularities, one can talk about

THE SINGULARITIES

instead of THE REGION OF ANALYTICITY.

- **Dyson's argument (1952)**

provides an effective qualitative criterion for non-analyticity of observables in terms of the system parameters.

A. General properties of solutions

How do physicists understand the Poincare theorem?

THE ANALYTIC *S* MATRIX

A BASIS FOR NUCLEAR DEMOCRACY

by

GEOFFREY F. CHEW

Формально указанная связь является отражением интуитивно понятной теоремы Пуанкаре, которая, грубо говоря, гласит следующее: если коэффициенты дифференциального уравнения аналитически зависят от некоторой величины, то и решения уравнения будут аналитическими функциями этой величины. Иными словами, Пуанкаре утверждает, что в теориях, основанных на дифференциальных уравнениях, сохраняется любая аналитичность, которую мы вводили в коэффициенты.

A. General properties of solutions

Poincare theorem on analyticity

How it works?

Example:

$$A_\varepsilon : \varepsilon \frac{du}{dx} = -u + x, \; 0 \leqslant x \leqslant 1; \quad u(0) = 1.$$

$$\frac{du}{dx} = \boxed{\frac{-u+x}{\varepsilon},}$$ r.h.s. has a simple pole at $\varepsilon = 0$.

Solution:

$$u_\varepsilon(x) = (1+\varepsilon)\boxed{\exp(-x/\varepsilon)} + x - \varepsilon.$$

The simple pole of ODE turns to the essential singularity of the solution

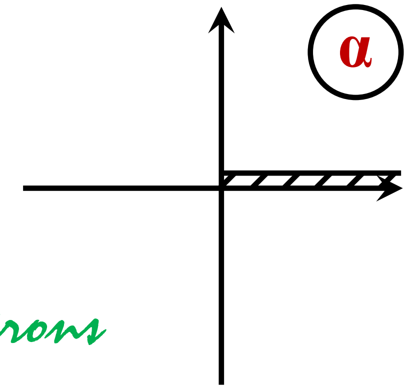
A. General properties of solutions

$\alpha > 0$: $0 < H_{QED}$

$\alpha < 0$: $-\infty < H_{QED}$

Dyson's argument

in QED



"a star" of positrons

"a star" electrons

BY

TUNNELING

If $\alpha < 0$: $E_{tot} < E_{vac} = 0$ for $N_{e+e-} \gtrsim 1/\alpha^{3/2}$ collapsing

$Gm^2 \longrightarrow \alpha$: "Chandrasekhar limit"

VACUUM IS UNSTABLE & $\alpha = 0$ A BRANCH POINT

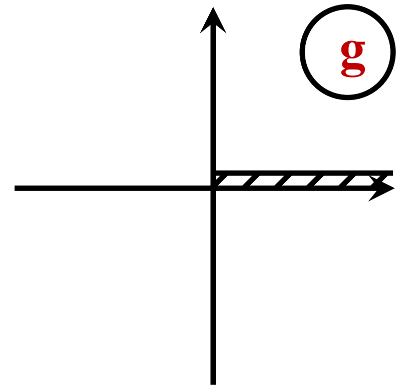
A. General properties of solutions

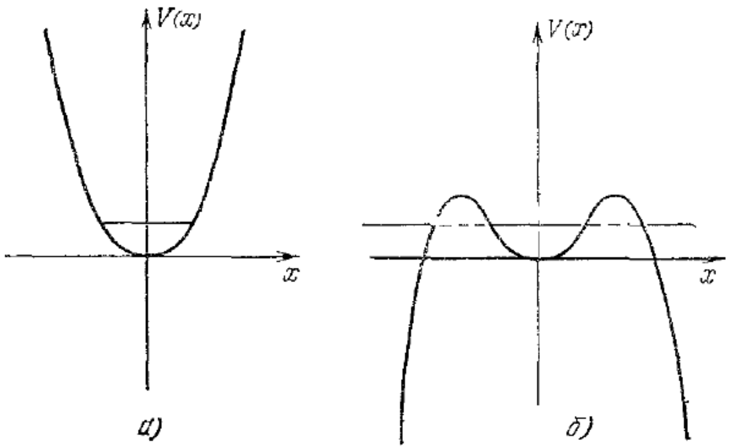
$g > 0$: $0 < H_4$

$g < 0$: $-\infty < H_4$

Dyson's argument

in Quantum mechanics





$$V(x) = m\omega^2 x^2/2 + g x^4$$

$$E_0 = E_0(g)$$

BOUND ($g > 0$) AND UNBOUND ($g < 0$)

HAMILTONIANS ARE QUALITATIVELY DIFFERENT

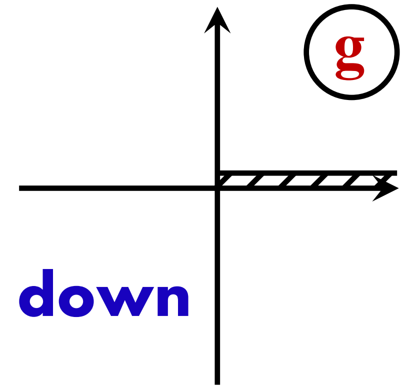
$g = 0$ is a branch point of E_0

А. В. Турбинер, УФН 144 35 (1984)

A. General properties of solutions

Dyson's argument

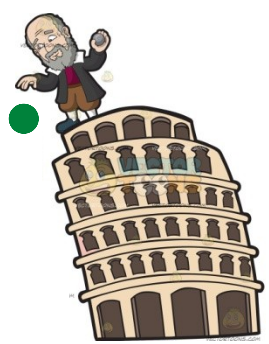
in Classical theory



 # 1. A school level example: Stone thrown down

$$\frac{mv^2}{2} = mgh,$$

$$v = \sqrt{2gh}.$$



g > 0

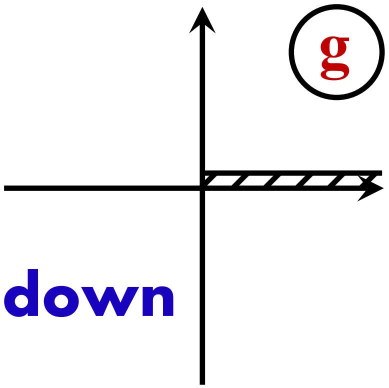
FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

g = 0 is a root branch point

A. General properties of solutions

Dyson's argument

in Classical theory



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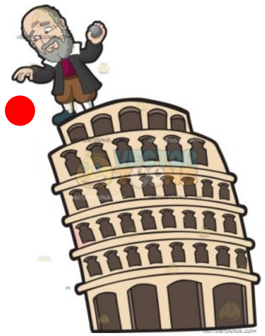
$$\frac{mv^2}{2} = mgh,$$

$$v = \sqrt{2gh}.$$



g < 0

FINITE (g > 0) AND INFINITE (g < 0) MOTIONS ARE QUALITATIVELY DIFFERENT

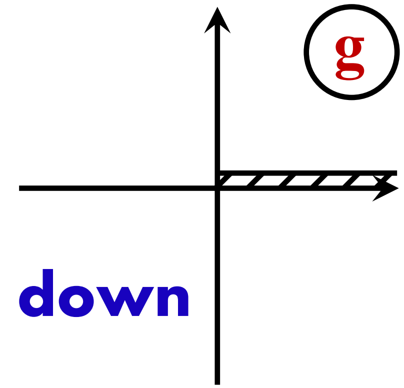
$\Updownarrow$

g = 0 is a root branch point

A. General properties of solutions

Dyson's argument

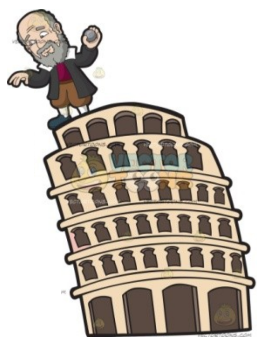
in Classical theory



 # 1. A school level example: Stone thrown down

$$\frac{mv^2}{2} = mgh,$$

$$v = \sqrt{2gh}.$$



FINITE (g > 0) AND INFINITE (g < 0)
MOTIONS ARE
QUALITATIVELY DIFFERENT

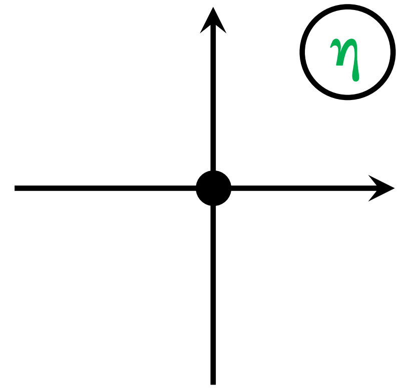
⬍

g = 0 is a root branch point

A. General properties of solutions

Dyson's argument

in Classical theory



2. Viscosity η in Navier–Stokes equation

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}\right)=-\nabla p+\boxed{\eta\Delta\boldsymbol{v}+\frac{\eta}{3}\nabla(\nabla\boldsymbol{v})}$$

Example: water flow in a tube:

$$v=\frac{\Delta p}{4\eta l}(R^2-r^2).$$

$\eta > 0$ IS OKEY

$\eta < 0$ is NOT PHYSICAL

TWO STRONGLY DIFFERENT CASES

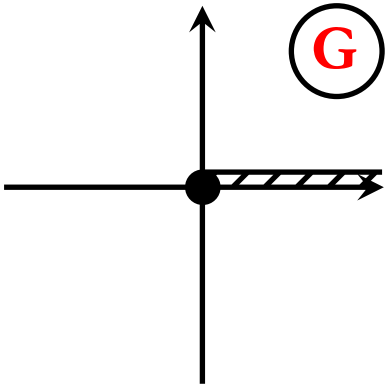
⇕

$\eta = 0$ is a singular point of v

A. General properties of solutions

Dyson's argument

in our case:

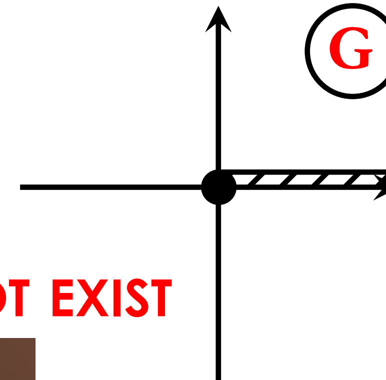


G > 0 STARS EXIST

A. General properties of solutions

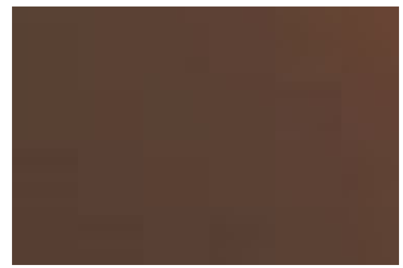
Dyson's argument

in our case:



$G > 0$ STARS EXIST

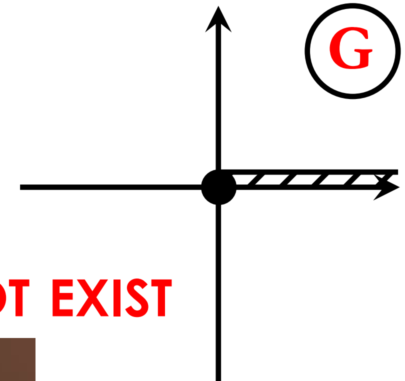
$G < 0$ STARS DO NOT EXIST



A. General properties of solutions

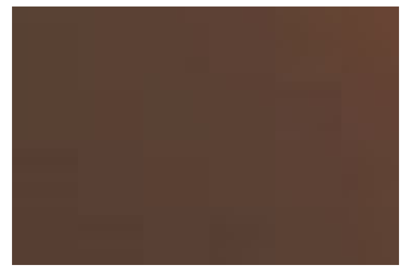
Dyson's argument

in our case:



$G > 0$ STARS EXIST

$G < 0$ STARS DO NOT EXIST



TWO CASES LOOK QUITE DIFFERENT

⬍

$G = 0$ is a singular point of solutions of hydrostatic equilibrium equations

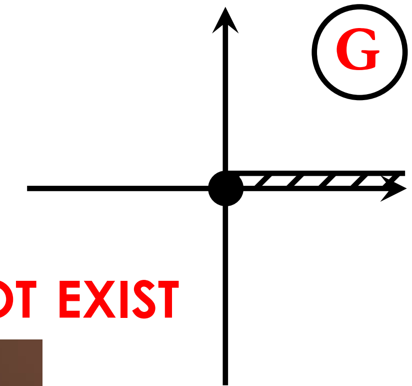
A. General properties of solutions

G < 0: 0 < H_G

G > 0: -∞ < H_G

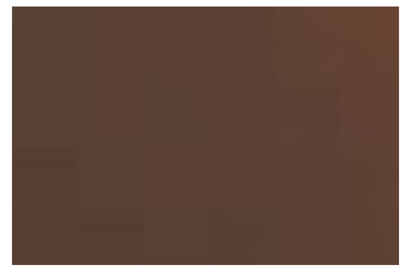
Dyson's argument

in our case:



G > 0 STARS EXIST

G < 0 STARS DO NOT EXIST



TWO CASES LOOK QUITE DIFFERENT

G = 0 is a singular point of solutions of hydrostatic equilibrium equations

MOREOVER,
RADIUS AND MASS OF STAR:

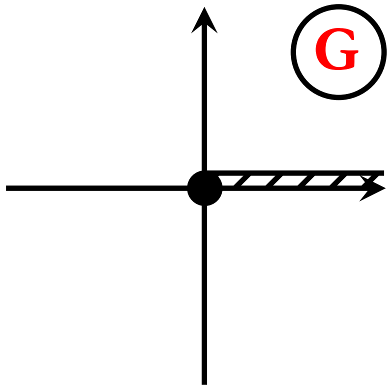
$$r_0^2 = \frac{Z\mu_{e0}}{4\pi G m_i (m_i + Z m_e) n_{i0}}$$

$$M_s = \int_0^{R_s} 4\pi r^2 (n_i m_i + n_e m_e) dr = -4\pi r_0^3 n_{i0} m_i x^2 \frac{d}{dx}(\theta_e + \Lambda_i \theta_i)\Big|_{x=x_b}$$

A. General properties of solutions

Dyson's argument

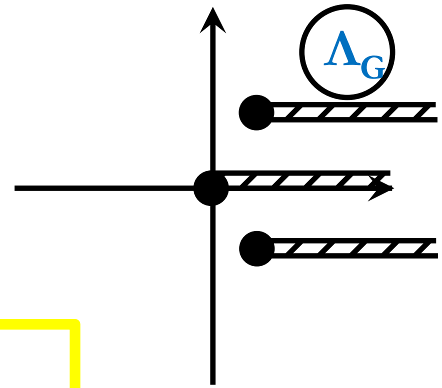
in our case:



The general solution should be sought in the class of functions that depend on Λ_G at $\Lambda_G = \infty$ in an irregular manner.

Exactly solvable model

B. Two-fluid model with unit polytropic indices



The above statements can be precisely formulated using the model $\eta_i = \eta_e = 1$. In terms of

$$\varphi_k \equiv x\theta_k(x)$$

the main system of HE equations become

$$\boxed{\begin{aligned} \varphi_e'' + \Lambda_i \varphi_i'' &= -(\varphi_i + \Lambda_m \Lambda_e \varphi_e) \\ \varphi_e'' - \Lambda_m \Lambda_i \varphi_i'' &= -\Lambda_G(\varphi_i - \Lambda_e \varphi_e) \end{aligned}}$$

We are looking for solutions in the form $\varphi_k = \alpha_k e^{\beta x}$ and obtain

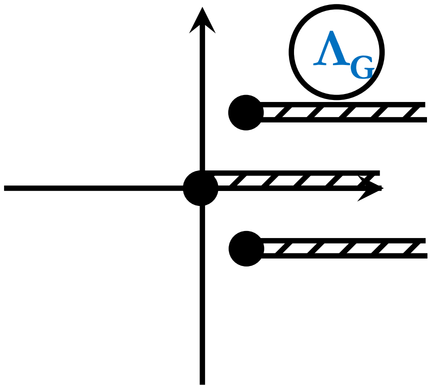
$$\begin{aligned} (\alpha_e + \Lambda_i \alpha_i)\beta^2 &= -(\alpha_i + \Lambda_m \Lambda_e \alpha_e), \\ (\alpha_e - \Lambda_m \Lambda_i \alpha_i)\beta^2 &= -\Lambda_G(\alpha_i - \Lambda_e \alpha_e). \end{aligned}$$

The eigenvalues are as follows:

$$\beta_\pm^2 = \frac{1}{2\Lambda_i(1+\Lambda_m)}\Big\{\Lambda_G(1+\Lambda_i\Lambda_e) - (1+\Lambda_m^2\Lambda_i\Lambda_e)$$
$$\pm \sqrt{\Lambda_G^2(1+\Lambda_i\Lambda_e)^2 - 2\Lambda_G\left[(1-\Lambda_m\Lambda_i\Lambda_e)^2 - \Lambda_i\Lambda_e(1+\Lambda_m)^2\right] + (1+\Lambda_m^2\Lambda_i\Lambda_e)^2}\Big\}$$

Exactly solvable model

B. Two-fluid model with unit polytropic indices



$$\beta_+^2 = \Lambda_\mathrm{G}\frac{1+\Lambda_\mathrm{i}\Lambda_\mathrm{e}}{\Lambda_\mathrm{i}(1+\Lambda_\mathrm{m})} + O(1),$$

$$\beta_-^2 = -\frac{\Lambda_\mathrm{e}(1+\Lambda_\mathrm{m})}{1+\Lambda_\mathrm{i}\Lambda_\mathrm{e}} + O(\Lambda_\mathrm{G}^{-1}).$$

The general solution: $\beta_1 = |\beta_+|$ & $\beta_2 = |\beta_-|$

$$\boxed{\varphi_\mathrm{k}(x) = \alpha_{\mathrm{k}1}\sinh(\beta_1 x) + \alpha_{\mathrm{k}2}\sin(\beta_2 x),}$$

with the constraint

$$\boxed{\alpha_{\mathrm{k}1}\beta_1 + \alpha_{\mathrm{k}2}\beta_2 = 1.}$$

to fulfill $\phi_k(0) = \phi_k(0)'' = 0$, $\phi_k(0)' = 1$.

NB: Для одноатомного идеального газа
показатель адиабаты $\gamma = 5/3$, у нас $\gamma = 1 + 1/\eta = 6/3$.

Exactly solvable model

B. Two-fluid model with unit polytropic indices

The general solution:

$$\boxed{\varphi_{\rm k}(x) = \alpha_{\rm k1}\sinh(\beta_1 x) + \alpha_{\rm k2}\sin(\beta_2 x),}$$

with the constraint

$$\boxed{\alpha_{\rm k1}\beta_1 + \alpha_{\rm k2}\beta_2 = 1.}$$

Regular (*Π ≡ 0*) solution: α_{1k} = 0,

$$\begin{aligned}(1+\Lambda_{\rm i})\beta_2^2 &= 1+\Lambda_{\rm m}\Lambda_{\rm e},\\ (1-\Lambda_{\rm m}\Lambda_{\rm i})\beta_2^2 &= \Lambda_{\rm G}(1-\Lambda_{\rm e}).\end{aligned}$$

IS POSSIBLE FOR

$$\Lambda_{\rm e} = \Lambda_{\rm e}^{\rm reg} = \frac{\Lambda_{\rm G}(1+\Lambda_{\rm i}) - (1-\Lambda_{\rm m}\Lambda_{\rm i})}{\Lambda_{\rm G}(1+\Lambda_{\rm i}) + \Lambda_{\rm m}(1-\Lambda_{\rm m}\Lambda_{\rm i})}.$$

We got a two-parameter set of the solutions.

CONCENTRATIONS AT THE CENTER OF THE STAR ARE ARBITRARY,

however, in an exponentially small vicinity of the regular solution.

__bulk charge

__bulk charge + electro/ionosphere

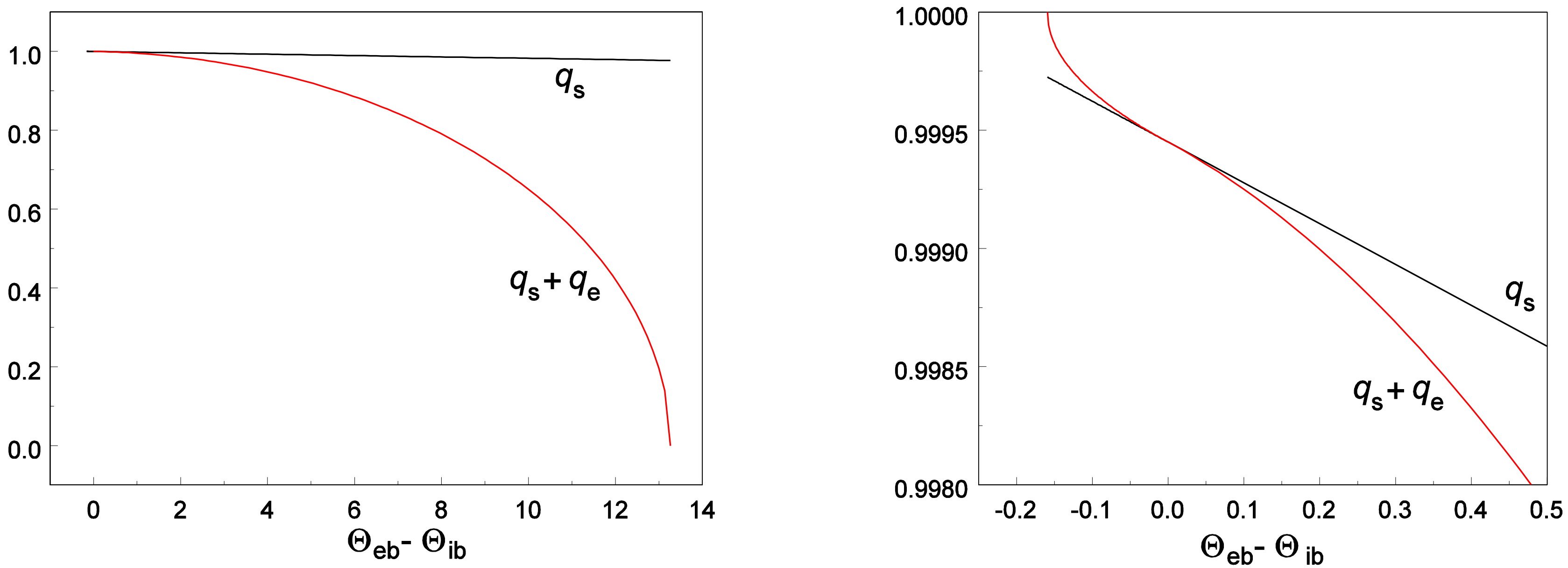


FIG. 4: (color online) The bulk charge q_s and the total charge $q_s + q_e$ as functions of the difference $\Theta_{eb} - \Theta_{ib}$ for a mixture of electrons and protons ($Z = A = 1$) and for $\Lambda_i = \Lambda_m$.

NB: $(q_s + q_e)^{max} = A/Z$

Exactly solvable model

B. Two-fluid model with unit polytropic indices

WHAT WE CAN LEARN:

■ **In the two-fluid model with unit polytropic indices, it is possible to explicitly construct the general solution describing charged stars with electro- and ionospheres.**

■ The guiding parameter of the problem is Λ_e

■ In an exponentially small neighborhood of $\Lambda_e = \Lambda_e^{reg}$, the electron and ion densities can be varied without restrictions.

■ The stellar envelope is sensitive to exponentially small deviations of Λ_e from Λ_e^{reg}.

General solution to the unconstrained hydrostatic equilibrium equations

The general solution is of the form:

$$\theta_{\rm k} = \theta_{\rm k0} + \chi_{\rm k},$$

regular irregular

In inner layers of the star $\chi_k \ll$

Small deviations in:

$$\triangle\Lambda_{\rm e} = \Lambda_{\rm e} - \Lambda_{\rm e}^{\rm reg},\ |\triangle\Lambda_{\rm e}| \ll 1$$

Linearization of the main ODE system gives

$$\boxed{\begin{aligned}
\triangle_x(\chi_{\rm e} + \Lambda_{\rm i}\chi_{\rm i}) &= -(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad +\Lambda_{\rm m}\Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} + \Lambda_{\rm m}\triangle\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}), \\
\triangle_x(\chi_{\rm e} - \Lambda_{\rm m}\Lambda_{\rm i}\chi_{\rm i}) &= -\Lambda_{\rm G}(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad -\Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} - \triangle\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}).
\end{aligned}}$$

$\chi_{\rm k}(0) = \chi'_{\rm k}(0) = 0$ ← initial conditions

General solution to the unconstrained hydrostatic equilibrium equations

A. Irregular component in the WKB approximation

Looking solutions in the form: $\boxed{\chi_{\mathrm{k}} = \chi_{\mathrm{k}}^{\mathrm{irr}} + \chi_{\mathrm{k0}}}$

General solution of <u>homogenious</u> equaiton

Particular solution of inhomogenious equaiton

Following the WKB method:

$$\chi_{\mathrm{k}}^{\mathrm{irr}}(x) = g_{\mathrm{k}}(x) \exp\left(\sqrt{\Lambda_{\mathrm{G}}} S(x)\right)$$

where k = i,e,

$$g_{\mathrm{k}}(x) = g_{\mathrm{k0}}(x) + \frac{g_{\mathrm{k1}}(x)}{\sqrt{\Lambda_{\mathrm{G}}}} + \frac{g_{\mathrm{k2}}(x)}{\Lambda_{\mathrm{G}}} + \dots.$$

- To the lowest order

$$S_{\pm}(x) = \pm \int_0^x \sqrt{\frac{\eta_{\mathrm{i}} \theta_{\mathrm{i0}}^{\eta_{\mathrm{i}}-1}(x') + \Lambda_{\mathrm{i}} \Lambda_{\mathrm{e}}^{\mathrm{reg}} \eta_{\mathrm{e}} \theta_{\mathrm{e0}}^{\eta_{\mathrm{e}}-1}(x')}{\Lambda_{\mathrm{i}}(1+\Lambda_{\mathrm{m}})}} dx'.$$

Finally (cf.: $1/\Lambda_G \leftarrow\rightarrow \hbar^2$)

$$\boxed{\chi_{\mathrm{k0}}^{\mathrm{irr}}(x) = C_{\mathrm{k0}} \frac{\sinh\left(\sqrt{\Lambda_{\mathrm{G}}} S_{+}(x)\right)}{x\sqrt{S'_{+}(x)}} + O(\Lambda_{\mathrm{G}}^{-\frac{1}{2}})}$$

General solution to the unconstrained hydrostatic equilibrium equations

B. Correction to the irregular component

$$\boxed{\theta_{\rm k} = \theta_{\rm k0} + \chi_{\rm k}, \qquad \chi_{\rm k} = \chi_{\rm k}^{\rm irr} + \chi_{\rm k0}}$$

Looking for a particular solution of the linearized inhomogeneous equation

$$\begin{aligned}
\Delta_x(\chi_{\rm e} + \Lambda_{\rm i}\chi_{\rm i}) &= -(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad + \Lambda_{\rm m}\Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} + \Lambda_{\rm m}\Delta\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}), \\
\Delta_x(\chi_{\rm e} - \Lambda_{\rm m}\Lambda_{\rm i}\chi_{\rm i}) &= -\Lambda_{\rm G}(\eta_{\rm i}\theta_{\rm i0}^{\eta_{\rm i}-1}\chi_{\rm i} \\
&\quad - \Lambda_{\rm e0}\eta_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}-1}\chi_{\rm e} - \Delta\Lambda_{\rm e}\theta_{\rm e0}^{\eta_{\rm e}}).
\end{aligned}$$

General solution to the unconstrained hydrostatic equilibrium equations

B. Correction to the irregular component

$$\boxed{\theta_{\mathrm{k}} = \theta_{\mathrm{k0}} + \chi_{\mathrm{k}}, \qquad \chi_{\mathrm{k}} = \chi_{\mathrm{k}}^{\mathrm{irr}} + \chi_{\mathrm{k0}}}$$

- Finally, the irregular part takes the form

$$\chi_{\mathrm{i}}(x) = \frac{C_{\mathrm{i0}}}{x\sqrt{S'_+(x)}}\sinh\left(\sqrt{\Lambda_{\mathrm{G}}}S_+(x)\right) - C_{\mathrm{i0}}\sqrt{\Lambda_{\mathrm{G}}S'_+(0)}\left[\theta_{\mathrm{i0}}(x) + x\frac{\partial\theta_{\mathrm{i0}}(x)}{\partial x}\frac{\eta_{\mathrm{i}}+\Lambda_{\mathrm{i}}}{2} + \Lambda_{\mathrm{i}}\frac{\partial\theta_{\mathrm{i0}}(x)}{\partial\Lambda_{\mathrm{i}}}(1+\Lambda_{\mathrm{i}})\right],$$

$$\chi_{\mathrm{e}}(x) = -\frac{C_{\mathrm{i0}}\Lambda_{\mathrm{i}}}{x\sqrt{S'_+(x)}}\sinh\left(\sqrt{\Lambda_{\mathrm{G}}}S_+(x)\right) + C_{\mathrm{i0}}\sqrt{\Lambda_{\mathrm{G}}S'_+(0)}\left[\Lambda_{\mathrm{i}}\theta_{\mathrm{e0}}(x) - \frac{\partial\theta_{\mathrm{e0}}(x)}{\partial x}\frac{\eta_{\mathrm{i}}+\Lambda_{\mathrm{i}}}{2} - \Lambda_{\mathrm{i}}\frac{\partial\theta_{\mathrm{e0}}(x)}{\partial\Lambda_{\mathrm{i}}}(1+\Lambda_{\mathrm{i}})\right]$$

The general solution of the hydrostatic equilibrium equations of a self-gravitating two-component charged fluid is known.

It was found only in 2018 through the efforts of Andrey Yudin and his collaborators in ITEP.

These equations remained unsolved for almost 70 years.

Three exactly solvable models of electrically charged stars in hydrostatic equilibrium

1. **Two-component charged fluid with polytrope indices $\eta_{\mathrm{i}} = \eta_{\mathrm{e}} = 1$ in Newton gravity**
   - Discussed in these lectures
   - P. Hund and M. K.-H. Kiessling, *Electrically nonneutral ground states of stars*, Phys. Rev. D 103, 043004 (2021)
2. **Incompressible two-component charged fluid with polytrope indices $\eta_{\mathrm{i}} = \eta_{\mathrm{e}} = 0$ in Newton gravity**
   - Appendix A in: M.I.K., D.K. Nadyozhin, A.V. Yudin, *Hydrostatic equilibrium of stars without electroneutrality constraint*, Phys. Rev. D 97, 083016 (2018).
3. **Incompressible three-component charged fluid (neutrons, protons, electrons) with polytrope indices $\eta_{\mathrm{n}} = \eta_{\mathrm{p}} = \eta_{\mathrm{e}} = 0$ in General Relativity**
   - N.I. Kramarev and A.V. Yudin, *The structure of relativistic stars consisting from an incompressible substance in the absence of a strict electroneutrality*, Letters to Astronomical Journal, 47, 646–656 (2021).

CONCLUSIONS

■ **A general solution of the hydrostatic equilibrium equations in the absence of local electroneutrality for a two-component charged liquid in the Newton theory of gravity and GRT is described.**

■ The problem belongs to the class of singular differential equations. The general solution is singular with respect to the gravitational constant G at G = 0, in the agreement with the Poincare theorem and Dyson's argument.

■ **We need to go beyond the framework of perturbation theory. To construct a general solution, one can apply the WKB method, well known to physicists in quantum mechanics.**

■ The electric charge of stars varies within Q_s = - 0.1 ÷ 100 C.

CONCLUSIONS

■ Three exactly solvable models of stars for polytrophic equations of state of matter are currently known.

■ The structure of electro- and ionosphere of stars are described.

UNSOLVED PROBLEMS

1. Possible applications of the developed techniques include:
2. Maximum charge and charge distribution in galaxy with a charged black hole in the center.
3. Jeans instability in charged giant molecular clouds.
4. Equilibrium equations for charged fluid in accretion disks.
5. Generalization of TOV-OB equations to rotating stars with magnetic field.
6. Electro- and ionospheres at finite temperature.


DÍKY ZA
VNÍMAVOST!

THANK YOU FOR ATTENTION!

Basic equations for N-component charged fluid

$$\frac{dm}{dr} = 4\pi r^2 \left[\rho + \frac{Q^2}{8\pi c^2 r^4}\right]$$

$$\frac{dQ}{dr} = 4\pi r^2 e^{\lambda/2} \sum q_i n_i \qquad e^{\lambda} = -g_{11} = \left[1 - \frac{2Gm}{rc^2}\right]^{-1}$$

from $\delta E/\delta n_i = 0$ under conditions $\delta N_i = 0$ one gets

$$\sum n_k \Bigg| \quad \frac{d\mu_k}{dr} = e^{\lambda/2}\frac{q_k Q}{r^2} - e^{\lambda}\frac{G\mu_k}{r^2 c^2}\left[m - \frac{Q^2}{2c^2 r} + \frac{4\pi r^3}{c^2}P\right].$$

Tolman–Oppenheimer–Volkoff equation for charged fluid

$$\frac{dP}{dr} = \frac{Q}{r^2}\frac{\sum q_i n_i}{\sqrt{1 - \frac{2Gm}{rc^2}}} - \frac{G\left(\rho + \frac{P}{c^2}\right)}{r\left(r - \frac{2Gm}{c^2}\right)}\left(m + \frac{4\pi r^3}{c^2}P - \frac{Q^2}{2rc^2}\right)$$

ρ is the energy density, $\quad$ i,k = 1,…,N

Q is the electric charge inside the sphere of radius r

m is the mass of the star inside the sphere of radius r

P is the pressure

T. Kodama and M. Yamada, Prog. Theor. Phys. 47, 444 (1972), E. Olson and M. Bailyn, Phys. Rev. D 12, 3030 (1975).