Theory of non-relativistic neutrinos (and non-relativistic dark matter)

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Motivation

- Non-relativistic quantum field theory sometimes simpler
- Condensed matter phenomena better understood in that framework
- Neutrinos become non-relativistic at late times
- Cold dark matter is also non-relatvistic
- Non-relativistic limit non-trivial for real fields

I. Scalars

Literature

- exact real scalar transformation: Namjoo, Guth, and Kaiser, “Relativistic corrections to nonrelativistic effective field theories”
- generalisation to arbitrary spacetime: Heyen and Floerchinger, “Real scalar field, the nonrelativistic limit, and the cosmological expansion”
- overview over different methods: Braaten, Mohapatra, and Zhang, “Classical nonrelativistic effective field theories for a real scalar field”

The starting point

We start with a known relativistic action $S[\phi]$ of a field ϕ, e.g. a complex scalar field without interaction:

$$S[\phi] = \int \mathrm{d}^4x \left\{(\partial_\mu \phi)(\partial^\mu \phi^*) - m|\phi|^2\right\}$$

$\rightarrow$ transform ϕ to a 'non-relativistic' field ψ such that the action can be expanded in momentum

The $c \to \infty$ limit

A simple ansatz for this is to start with the equation of motion

$$\frac{1}{c^2}\partial_t^2\phi - \vec{\nabla}^2\phi + m^2c^2\phi = 0$$

and assume that the energy of particle is close to its rest energy. The resulting phase can be factored out as

$$\phi = \frac{1}{\sqrt{2m}}e^{-imc^2t}\psi$$

Non-relativistic is here defined by the particle energy being close to the positive mass pole of its propagator. The equation of motion becomes

$$\frac{1}{2mc^2}\partial_t^2\psi - i\partial_t\psi - \frac{1}{2m}\vec{\nabla}^2\psi = 0.$$

The $c \to \infty$ limit

In the limit $c \to \infty$ the first term drops out and the resulting equation of motion for ψ is the free Schroedinger equation.

$$-i\partial_t \psi - \frac{\vec{\nabla}^2}{2m}\psi = 0 \quad .$$

This equation is first order in time derivatives while the Klein-Gordon equation we started with is second order. The degrees of freedom we lost in taking the limit is the information about antiparticles.

Real scalar fields

The $c \to \infty$ limit fails for real scalar, where all terms would oscillate infinitely fast. Starting at the action again

$$S = \int \mathrm{d}^4x \left\{ \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 \right\}$$

we can look at it from a Hamiltonian perspective

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{1}{2}m^2\phi^2$$

with the equations of motion

$$\dot{\phi} = \pi, \qquad \dot{\pi} = (\vec{\nabla}^2 - m^2)\phi.$$

NGK Transformation

Transforming ϕ and π into a single complex field with independent real and imaginary part would keep the same number of degrees of freedom.
Namjoo, Guth and Kaiser found such a transformation for real scalar fields[1]

$$\psi = \sqrt{\frac{m}{2}} e^{imt} \mathcal{P}^{1/2} \left(\phi + \frac{i}{m} \mathcal{P}^{-1} \pi \right) \quad ,$$

$$\mathcal{P} = \sqrt{1 - \frac{\vec{\nabla}^2}{m^2}}$$

Where the non-local operator $\mathcal{P}$ can be understood via its Taylor expansion

$$\mathcal{P} = \sqrt{1 - \frac{\vec{\nabla}^2}{m^2}} = 1 - \frac{\vec{\nabla}^2}{2m^2} - \frac{(\vec{\nabla}^2)^2}{8m^4} + \dots$$

or via its momentum space representation

$$\mathcal{P} = \sqrt{1 + \frac{\vec{p}^2}{m^2}} = m^{-1} E_{\vec{p}}$$

[1] Namjoo, Guth, and Kaiser, "Relativistic corrections to nonrelativistic effective field theories".

NGK Transformation

This transformation is canonical and can be inverted as

$$\begin{aligned}\phi =&\frac{1}{\sqrt{2m}}\mathcal{P}^{-\frac{1}{2}}\left(e^{-imt}\psi + e^{+imt}\psi^*\right),\\ \pi =& -i\sqrt{\frac{m}{2}}\mathcal{P}^{\frac{1}{2}}\left(e^{-imt}\psi - e^{+imt}\psi^*\right).\end{aligned}$$

At this point no information has been lost, the two descriptions are equivalent. The equation of motion of the transformed field is

$$i\partial_t\psi = m(\mathcal{P}-1)\psi$$

NGK Transformation

The point at which we lose information lies in the expansion of this equation of motion in p^2/m^2. To lowest order this is the Schroedinger equation, but higher orders can be systematically obtained.

$$i\partial_t\psi = \left[-\frac{\vec{\nabla}^2}{2m} - \frac{(\vec{\nabla}^2)^2}{8m^3} + \dots\right]\psi$$

The Lagrangian would be

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - \psi^* m(\mathcal{P}-1)\psi \\ &\approx \frac{i}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) + \psi^*\frac{\vec{\nabla}^2}{2m}\psi\end{aligned}$$

Adding interactions

Now we go to an interacting theory

$$S = \int \mathrm{d}^4x \left\{ \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}$$

The additional term in the action adds a term to the equation of motion

$$i\partial_t\psi = m(\mathcal{P}-1)\psi + \frac{\lambda}{4!m^2}\mathcal{P}^{-1/2}e^{imt}(P^{-1/2}e^{-imt}\psi + P^{-1/2}e^{imt}\psi^*)^3$$

which contains fast oscillating terms that we expect to average out over time.

Adding interactions

In order to get rid of the oscillating terms, we define auxiliary quantities and expand them in frequency modes which are multiples of the mass

$$G = \mathcal{P}^{-1/2} e^{imt} (P^{-1/2} e^{-imt} \psi + P^{-1/2} e^{imt} \psi^*)^3 = \sum_{\nu=-\infty}^{\infty} G_\nu e^{i\nu m t}$$

$$\psi = \sum_{\nu=-\infty}^{\infty} \psi_\nu e^{i\nu m t}$$

$$\Psi = \mathcal{P}^{-1/2} \psi = \sum_{\nu=-\infty}^{\infty} \Psi_\nu e^{i\nu m t}$$

We can further identify

$$G_\nu = \mathcal{P}^{-1/2} \sum_{\mu,\mu'=-\infty}^{\infty} \Big\{ \Psi_\mu \Psi_{\mu'} \Psi_{2+\nu-\mu-\mu'} + \Psi^*_\mu \Psi^*_{\mu'} \Psi^*_{4-\nu-\mu-\mu'} + 3\Psi_\mu \Psi_{\mu'} \Psi^*_{\mu+\mu'-\nu} + 3\Psi^*_\mu \Psi^*_{\mu'} \Psi_{\nu-2+\mu+\mu'} \Big\}.$$

Adding interactions

We can use this splitting to write down equations of motions for the frequency modes

$$i\dot{\psi}_\nu - \nu m \psi_\nu = m(\mathcal{P} - 1)\psi_\nu + \frac{\lambda}{4! m^2} G_\nu$$

The mode we will be looking at is the ground mode ψ_0. In order to obtain its dynamics we assume several quantities to be small. This includes all non-ground modes Ψ_ν and their time derivatives $\dot{\Psi}_\nu / \Psi_\nu$, the interaction strength λ and all appearances of $\vec{\nabla}^2$.

Adding interaction

Expanding to second order in small quantities, we will replace every non-ground mode by its truncated inverted equation of motion

$$\Psi_{\nu\neq 0} = \frac{\lambda(1-\nu-\mathcal{P})^{-1}\mathcal{P}^{-1/2}}{4!m^3}G_\nu$$

Following this scheme and expanding all the $\mathcal{P}$ operators leads to

$$\begin{aligned} i\dot{\psi}_s \approx & -\frac{1}{2m}\vec{\nabla}^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s - \frac{1}{8m^3}\nabla^4\psi_s \\ & + \frac{\lambda}{32m^4}\left[\psi_s^2\vec{\nabla}^2\psi_s^* + 2|\psi_s|^2\vec{\nabla}^2\psi_s + \vec{\nabla}^2(|\psi_s|^2\psi_s)\right] \\ & - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s. \end{aligned}$$

Adding interaction

Translated back into an effective Lagrangian, this becomes

$$\begin{aligned}\mathcal{L}_{\text{eff}} =& \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2m}(\vec{\nabla}\psi_s)(\vec{\nabla}\psi_s^*) - \frac{\lambda}{16m^2}|\psi_s|^4 \\ &+ \frac{1}{8m^3}(\vec{\nabla}^2\psi_s)(\vec{\nabla}^2\psi_s^*) - \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\vec{\nabla}^2\psi_s + \psi_s\vec{\nabla}^2\psi_s^*) \\ &+ \frac{17\lambda^2}{9\cdot 2^8 m^5}|\psi_s|^6.\end{aligned}$$

which contains relativistic corrections, higher order interactions and derivative interactions.

Going to a cosmological setting

In cosmological problems we cannot always assume spacetime to be Minkowskian. We will consider a scenario of expanding spacetime using the Friedmann-Lemâitre-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\vec{x}^2$$
$$(g_{\mu\nu}) = \mathsf{diag}(-1, a^2, a^2, a^2)$$

The action of a free real scalar on this background is

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L} = \int \mathrm{d}^4x\,a(t)^3\mathcal{L}$$
$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2$$
$$= \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}a(t)^{-3}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

$c \to \infty$ *in FLRW spacetime*

The equation of motion of a free complex scalar in FLRW spacetime is

$$\frac{1}{c^2}\partial_t^2\phi + \frac{3H}{c^2}\partial_t\phi - a^{-2}\vec{\nabla}^2\phi + m^2c^2\phi = 0$$

where $H = \dot{a}/a$ is the Hubble rate. As in the Minkowskian case, we can now replace transform the field according to

$$\phi = \frac{1}{\sqrt{2m}}e^{-imc^2t}\psi$$

and take the limit $c \to \infty$. We end up with a slightly modified equation of motion

$$-i\partial_t\psi - \frac{3}{2}iH\psi - \frac{1}{2a^2m}\vec{\nabla}^2\psi = 0.$$

The expanding spacetime scales the spatial derivatives and introduces a damping term.

Covariantly non-relativistic?

Velocities being non-relativistic is a frame-dependent statement. We define the frame in which the system is non-relativistic by chosing a timelike unit vector u^μ as its time direction.

$$u^\mu u_\mu = -1$$

We define the projector perpendicular to u^μ as

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

With this frame of reference a velocity v^μ being non-relativistic means

$$\Delta_{\mu\nu} v^\mu v^\nu \ll (u_\mu v^\mu)^2$$

Building the transformation

Staying close to the NGK transformation we want a generalized transformation that is:

- linear
- local in the time-direction defined by u^μ
- an exact transformation to $iu^\mu\partial_\mu\psi = \hat{O}\psi$
- canonical (up to a constant factor)

Covariant formalism

With these conditions for the transformation our ansatz becomes

$$\psi = \alpha(\phi + i\gamma u^\mu \nabla_\mu \phi)$$

where α and γ are both differential operators that do not contain $u^\mu \nabla_\mu$. For Minkwoski spacetime, we recover the NGK transformation by chosing

$$u^\mu = (1, 0, 0, 0)^T$$
$$\alpha = \sqrt{\frac{m}{2}} \mathcal{P}^{1/2} e^{imt}$$
$$\gamma = (m\mathcal{P})^{-1}$$

Covariant formalism

The equation of motion for the relativistic field ϕ is

$$-g^{\mu\nu}\nabla_\mu\partial_\nu\phi + m^2\phi = 0$$

With this we can calculate the time derivative of the transformed field as

$$\begin{aligned} u^\mu\partial_\mu\psi = &[u^\mu\partial_\mu\alpha + i\beta(-m^2 + \nabla_\mu\Delta^{\mu\nu}\partial_\nu)]\phi \\ &+ [\alpha + iu^\nu\partial_\nu\beta - i\beta\nabla_\mu u^\mu](u^\mu\partial_\mu\phi) \end{aligned}$$

where we replaced the second order time derivative with the equation of motion of ϕ. In order for this to be of the desired form, the first bracket has to be proportional to α and the second has to be the same factor multiplied by $i\alpha\gamma$.

Covariant formalism

This leads to a differential equation for γ

$$u^\nu \partial_\nu \gamma + iB\gamma^2 - (\nabla_\mu u^\mu)\gamma - i = 0$$
$$B = (m^2 - \nabla_\mu \Delta^{\mu\nu} \partial_\nu)$$

Where there derivatives in spatial directions are something that γ can explicitly depend on, while the time derivatives are not. This fixes the equation of motion of the transformed field to

$$u^\mu \partial_\mu \psi = [\alpha^{-1} u^\mu \partial_\mu \alpha - iB\gamma]\psi$$

Covariant formalism

The transformation is canonical (up to a factor) if we can find a generating function $F(\phi, \mathrm{Im}(\psi))$ with

$$u^\mu \partial_\mu \phi = \frac{\partial F}{\partial \phi} \ , \ \ \mathrm{Re}(\psi) = \frac{\partial F}{\partial \, \mathrm{Im}(\psi)}$$

Such a function can be found

$$F(\phi, \mathrm{Im}(\psi)) = - \, \mathrm{Im}(\psi) \left(\frac{\mathrm{Im}(\alpha\gamma)}{\mathrm{Re}(\alpha\gamma)} \right) \mathrm{Im}(\psi) - \frac{1}{2} \phi \left(\frac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha\gamma)} \right) \phi$$
$$+ \, \mathrm{Im}(\psi) \left(\frac{1}{\mathrm{Re}(\alpha\gamma)} \right) \phi.$$

only if we assume the relation

$$|\alpha|^2 = (2 \, \mathrm{Re}(\gamma))^{-1}$$

Covariant formalism

In order to further simplify the equation we can look at the real part of the differential equation for γ which (assuming all eigenvalues of B are real) gives us

$$B\,\mathrm{Im}(\gamma) = \frac{u^\mu \partial_\mu \mathrm{Re}(\gamma)}{2\,\mathrm{Re}(\gamma)} - \frac{1}{2}\nabla_\mu u^\mu$$

The real part of the right hand side of the transformed equation of motion also contains

$$\mathrm{Re}\left(\frac{u^\mu \partial_\mu \alpha}{\alpha}\right) = \frac{u^\mu \partial_\mu |\alpha|}{|\alpha|} = -\frac{1}{2}\frac{u^\mu \partial_\mu \mathrm{Re}(\gamma)}{\mathrm{Re}(\gamma)}.$$

The two identities can be used to simplify the equation of motion to

$$i\left[u^\mu \partial_\mu \psi + \frac{1}{2}(\nabla_\mu u^\mu)\psi\right] = -\left[u^\mu \partial_\mu \arg(\alpha) - B\,\mathrm{Re}(\gamma)\right]\psi.$$

Covariant formalism

The Lagrangian belonging to this equation of motion is exactly what we would expect for a non-relativistic theory

$$\mathcal{L} = \frac{i}{2}\left[(u^\mu \partial_\mu \psi)\psi^* - \psi(u^\mu \partial_\mu \psi^*)\right] + \psi^* \left[u^\mu \partial_\mu \arg(\alpha) - B\,\mathrm{Re}(\gamma)\right]\psi$$

In Minkowski spacetime with $u^\mu = (1,0,0,0)^T$, $\arg(\alpha) = mt$ and $B\,\mathrm{Re}(\gamma) = m\mathcal{P}$ we return to the Lagrangian of the NGK transformed scalar.

$$\mathcal{L} = \frac{i}{2}\left[(u^\mu \partial_\mu \psi)\psi^* - \psi(u^\mu \partial_\mu \psi^*)\right] - \psi^* m(\mathcal{P} - 1)\psi$$

Application to FLRW

Going back to specifically FLRW spacetime, we now have another scale added in H. For simplicity we will consider late times

$$\frac{H}{m} \ll 1.$$

The three choices left for us to make are the choice of frame, the initial condition of the differential equation for γ and the phase of α. For the first the simplest choice is

$$u^\mu = (1, 0, 0, 0)^T$$

With this the spatial projector and the differential operators become

$$\begin{aligned} \Delta^{\mu\nu} &= a^{-2}\text{diag}(0, 1, 1, 1) \\ u^\mu \partial_\mu &= \partial_t \\ B &= m^2 - \nabla_\mu \Delta^{\mu\nu} \partial_\nu = m^2 - a^{-2}\vec{\nabla}^2 \end{aligned}$$

Application to FLRW

With this the differential equation for γ becomes

$$u^\nu \partial_\nu \gamma + i(m^2 - a^{-2}\vec{\nabla}^2)\gamma^2 - (\nabla_\mu u^\mu)\gamma - i = 0$$

We expand γ in orders of H/m and chose as our initial condition that the lowest order matches with the NGK transformation in the limit $a \to 1$

$$\gamma = \frac{1}{m}\left(\mathcal{P}_a^{-1} - \frac{iH}{2m}(\mathcal{P}_a^{-4} + 2\mathcal{P}_a^{-2}) + O\left(\left(\frac{H}{m}\right)^2\right)\right)$$

with a modified differential operator

$$\mathcal{P}_a = \sqrt{1 - \frac{\vec{\nabla}^2}{a^2 m^2}}$$

Application to FLRW

The phase of α can again be chosen as $\arg(\alpha) = mt$

$$\alpha = \sqrt{\frac{m}{2}} e^{imt} \mathcal{P}_a^{1/2}$$

The full transformation is

$$\psi = \sqrt{\frac{m}{2}} e^{imt} \mathcal{P}_a^{1/2} \left(\phi + \frac{i}{m} \left(\mathcal{P}_a^{-1} - \frac{iH}{2m} (\mathcal{P}_a^{-4} + 2\mathcal{P}_a^{-2}) \right) u^\mu \partial_\mu \phi \right)$$

The equation of motion for the transformed field then becomes

$$i \left(\dot{\psi} + \frac{3}{2} H \psi \right) = m(\mathcal{P}_a - 1)\psi \approx -\frac{\vec{\nabla}^2}{2a^2 m} \psi$$

which to lowest order matches with the $c \to \infty$ limit.

Effective Lagrangian for ϕ^4 in FLRW spacetime

Adding a four-point interaction changes the Lagrangian to

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

and adds a term to the equation of motion

$$i\left(\dot{\psi} + \frac{3}{2}H\psi\right) = m(\mathcal{P}_a - 1)\psi + \frac{\lambda}{3!}\alpha\gamma(\alpha^*\gamma^*\psi + \alpha\gamma\psi^*)^3$$

which we expand in the same frequency modes as for the NGK transformation.

Effective Lagrangian for ϕ^4 in FLRW spacetime

The difference here is that we also consider H/m small in addition to $\frac{\vec{p}^2}{m^2}$, λ and $\frac{\dot{\psi}_\nu}{m\psi_\nu}$.
The resulting effective Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} =& \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2a^2m}(\vec{\nabla}\psi_s)(\vec{\nabla}\psi_s^*) - \frac{\lambda}{16m^2}|\psi_s|^4 \\
&+ \frac{1}{8a^4m^3}(\vec{\nabla}^2\psi_s)(\vec{\nabla}^2\psi_s^*) - \frac{\lambda}{32a^2m^4}|\psi_s|^2(\psi_s^*\vec{\nabla}^2\psi_s + \psi_s\vec{\nabla}^2\psi_s^*) \\
&+ \frac{17\lambda^2}{9\cdot 2^8m^5}|\psi_s|^6 - i\frac{3\lambda H}{32m^3}|\psi_s|^4.
\end{aligned}
$$

which contains additional factors of a scaling each spatial derivative and a new term proportional to the Hubble rate H.

Complex scalar fields

A free complex scalar on a general background metric has the Lagrangian

$$\mathcal{L} = -g^{\mu\nu}(\partial_\mu \Phi^*)(\partial_\nu \Phi) - m^2 |\Phi|^2$$

If we transform the complex scalar according to

$$\Psi_1 = \alpha(\Phi + i\gamma u^\mu \nabla_\mu \Phi) = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2),$$
$$\Psi_2 = \alpha(\Phi^* + i\gamma u^\mu \nabla_\mu \Phi^*) = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

the Lagrangian becomes

$$\mathcal{L} = \sum_{n=1}^{2} \Big\{ \frac{i}{2}((u^\mu \partial_\mu \Psi_n)\Psi_n^* - \Psi_n(u^\mu \partial_\mu \Psi_n^*)) + \Psi_n^* \left(u^\mu \partial_\mu \arg(\alpha) - B\,\mathrm{Re}(\gamma)\right) \Psi_n \Big\}$$

which contains just two identical independent non-relativistic fields representing particles and antiparticles.

II. Features of non-relativistic theories

Symmetries of non-relativistic quantum field theories

Consider *Gross-Pitaevskii* action

$$S = \int dt d^{d-1}x \left\{ \varphi^*(t,\vec{x}) \left[i\partial_t + \frac{\vec{\nabla}^2}{2m} - V(t,\vec{x}) \right] \varphi(t,\vec{x}) - \frac{\lambda}{2} \varphi^{*2}(t,\vec{x})\varphi^2(t,\vec{x}) \right\}$$

Galilei transformations as non-relativistic limit of Poincare transformations

$$(t, x^j) \to (t + a, R^j{}_k x^k + v^j t + b^j)$$

Composition law

$$(R_2, a_2, \vec{v}_2, \vec{b}_2) \circ (R_1, a_1, \vec{v}_1, \vec{b}_1) = (R_2 R_1, a_2 + a_1, \vec{v}_2 + R_2\vec{v}_1, \vec{b}_2 + R_2\vec{b}_1 + \vec{v}_2 a_1)$$

Rotations

Rotations are realized as in the relativistic case with hermitian generators $J_j = \frac{1}{2}\epsilon_{jkl}M_{kl}$. When acting on a scalar field we have $M_{kl} = \mathcal{M}_{kl}$ where

$$\mathcal{M}_{kl} = -i(x_k\partial_l - x_l\partial_k).$$

The action on a scalar field is simply

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = \varphi(t, R^{-1}\vec{x}).$$

For non-relativistic spinor or tensor fields, this is supplemented by the appropriate generator acting on the "internal" representation of the field

Translations in space and time

Translations in space and time are also implemented as in the relativistic case. They are generated by

$$P_0 = -H = -i\frac{\partial}{\partial t}, \qquad P_j = -i\frac{\partial}{\partial x^j}.$$

The action of a finite group element on the scalar field is

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \varphi(t-a, \vec{x}-\vec{b}).$$

For an infinitesimal transformation this becomes

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \Big(1 - idaP_0 - idb^j P_j\Big)\,\varphi(t,\vec{x}).$$

The last equation is for an infinitesimal transformation. Eigenfunctions are plane waves, $e^{-i\omega t + i\vec{p}\vec{x}}$, as usual.

Galilei boosts

Galilei boosts are realized in a somewhat non-trivial way

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = e^{im\vec{v}\vec{x} - i\frac{m\vec{v}^2 t}{2}} \varphi(t,\vec{x}-\vec{v}t)$$

Infinitesimal transformation

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \left(1 - idv^j K_j\right) \varphi(t,\vec{x}),$$

where the boost generator is

$$K_j = -mx_j - it\frac{\partial}{\partial x^j}.$$

In particular, we see that this depends through the first term on the particle mass m.

Lie algebra

Commutation relations contain particle mass m as *central charge*

$$
\begin{aligned}
&[P_j, P_k] = [P_0, P_k] = [K_j, K_k] = 0, \\
&[J_j, J_k] = i\epsilon_{jkl} J_l, \qquad [J_j, P_0] = 0 \\
&[J_j, P_k] = i\epsilon_{jkl} P_l, \qquad [J_j, K_k] = i\epsilon_{jkl} K_l, \\
&[K_j, P_0] = -iP_j, \qquad [K_j, P_k] = -im\delta_{jk}.
\end{aligned}
$$

Galilei group realized as *projective representation*.
Superselection rule: linear superpositions of states with different mass m are not allowed.
Alternatively introduce *mass operator* M as element of the Lie algebra

$$
\begin{aligned}
&[K_j, P_k] = -iM\delta_{jk}, \\
&[M, P_0] = [M, P_j] = [M, J_j] = [M, K_j] = 0.
\end{aligned}
$$

Mass operator is in the *center* of the algebra.

Galilei covariant derivative

Consider transformation of time derivative term

$$\varphi^*(t,\vec{x})\,[i\partial_t]\,\varphi(t,\vec{x}) \to \varphi^*(t,\vec{x}-\vec{v}t)\left[i\partial_t + \frac{m\vec{v}^2}{2} - i\vec{v}\,\vec{\nabla}\right]\varphi(t,\vec{x}-\vec{v}t).$$

Consider also the spatial derivative term

$$\varphi^*(t,\vec{x})\left[\frac{\vec{\nabla}^2}{2m}\right]\varphi(t,\vec{x}) \to \varphi^*(t,\vec{x}-\vec{v}t)\left[\frac{\vec{\nabla}^2}{2m} - \frac{m\vec{v}^2}{2} + i\vec{v}\,\vec{\nabla}\right]\varphi(t,\vec{x}-\vec{v}t).$$

Neither the time derivative term nor the spatial derivative terms are Galilei covariant by themselves. However, their combination is. The combination

$$\mathscr{D} = i\partial_t + \frac{\vec{\nabla}^2}{2m},$$

is acting as a *covariant derivative* with respect to Galilei boost transformations

$$\mathscr{D}\varphi(t,\vec{x}) \to \mathscr{D}e^{im\vec{v}\vec{x}-i\frac{m\vec{v}^2 t}{2}}\varphi(t,\vec{x}-\vec{v}t) = e^{im\vec{v}\vec{x}-i\frac{m\vec{v}^2 t}{2}}\mathscr{D}\varphi(t,\vec{x}-\vec{v}t).$$

Global $U(1)$ transformations

The action has another symmetry, namely under global $\mathrm{U}(1)$ transformations,

$$\varphi(t,\vec{x}) \to e^{i\alpha}\varphi(t,\vec{x}), \qquad \varphi^*(t,\vec{x}) \to e^{-i\alpha}\varphi^*(t,\vec{x}).$$

Consequence is the conservation law for particle number

$$\partial_t n + \vec{\nabla}\vec{n} = 0.$$

Here $n = \varphi^*\varphi$ is the particle number density and

$$\vec{n} = -\frac{i}{m}\left(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right),$$

is the corresponding current.

Time-dependent $U(1)$ transformations

Consider time dependent $\mathrm{U}(1)$ transformations,

$$\varphi(t,\vec{x}) \to e^{i\alpha(t)}\varphi(t,\vec{x}), \qquad \varphi^*(t,\vec{x}) \to e^{-i\alpha(t)}\varphi^*(t,\vec{x}).$$

Time derivative term transforms as

$$[i\partial_t]\,\varphi \to [i\partial_t]\,e^{i\alpha(t)}\varphi = e^{i\alpha(t)}\,[-\partial_t\alpha(t) + i\partial_t]\,\varphi.$$

Can be compensated by a change in the external potential,

$$V(t,\vec{x}) \to V(t,\vec{x}) - \partial_t\alpha(t).$$

The extended combination

$$\mathscr{D} = i\partial_t + \frac{\vec{\nabla}^2}{2m} - V(t,\vec{x}),$$

is a covariant derivative for both Galilei boost and time-dependent $\mathrm{U}(1)$ transformations.

Scaling transformations

Theories without interaction or at renormalization group fixed points have more symmetries. Non-relativistic scaling

$$(t, \vec{x}) \to \left(\lambda^2 t, \lambda \vec{x}\right).$$

Dilation transformation acting on the field,

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = \lambda^{(1-d)/2} \varphi(\lambda^{-2} t, \lambda^{-1} \vec{x}).$$

For an infinitesimal transformation $\lambda = 1 + c$ this can be written as

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = (1 - icD)\, \varphi(t, \vec{x}),$$

with the dilatation operator acting acting on the scalar fields given by

$$D = -i \left(x^j \frac{\partial}{\partial x^j} + 2t \frac{\partial}{\partial t} \right) - i \frac{d-1}{2}.$$

Time-dependent scaling or "Schrödinger expansion"

Another interesting symmetry of the free Schrödinger equation with respect to the time-dependent scaling

$$(t, \vec{x}) \to (t', \vec{x}') = \left(\frac{t}{1+ft}, \frac{\vec{x}}{1+ft} \right).$$

As an infinitesimal transformation this reads

$$(t, \vec{x}) \to (t', \vec{x}') = (t - dft^2, \vec{x} - dft\vec{x}).$$

Infinitesimal transformation of the fields

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = (1 - idfC)\,\varphi(t, \vec{x}),$$

with the generator for a *special conformal transformation*

$$C = -i \left(t^2 \frac{\partial}{\partial t} + tx^j \frac{\partial}{\partial x^j} \right) - it\frac{d-1}{2} - \frac{1}{2} m\vec{x}^2.$$

Full Schrödinger algebra

There are 12 generators in total (not counting the central charge)

- 3 rotations
- 3 Galilei boosts
- 3 translations in space
- 1 translation in time
- 1 dilatation
- 1 special conformal transformation

Dilations and Special conformal symmetry transformations are often broken by interaction terms.

Additional commutation relations

$$
\begin{aligned}
&[D, J_j] = 0, \qquad [D, P_j] = iP_j, \qquad [D, P_0] = 2iP_0, \\
&[D, K_j] = -iK_j, \qquad [D, M] = 0, \\
&[C, J_j] = 0, \qquad [C, K_j] = 0, \qquad [C, P_j] = iK_j, \\
&[C, P_0] = iD, \qquad [C, D] = 2iC.
\end{aligned}
$$

Effective potential

Write the action as

$$S = \int dt d^3x \left\{ \varphi^* \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \varphi - V(\varphi^* \varphi) \right\},$$

with microscopic potential as a function of $\rho = \varphi^* \varphi$,

$$V(\rho) = V_0 \rho + \frac{\lambda}{2} \rho^2 = -\mu \rho + \frac{\lambda}{2} \rho^2 .$$

At non-vanishing density one has $V_0 = -\mu$, where μ is the chemical potential.
For $\mu > 0$ the minimum of the effective potential is at $\rho_0 > 0$.

If the solution $\varphi(x) = \phi_0$ is homogeneous (constant in space and time), it must correspond to a minimum of the effective potential,

$$V'(\rho_0) = -\mu + \lambda\rho_0 = 0,$$

or

$$\phi_0 = \sqrt{\rho_0} = \sqrt{\frac{\mu}{\lambda}}.$$

Breaks the global $\mathrm{U}(1)$ symmetry spontaneously: Bose-Einstein condensation.

Bogoliulov excitations

Study small perturbations around the homogeneous field value ϕ_0,

$$\varphi(x) = \phi_0 + \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\,\phi_2(x)\right],$$

with real fields $\phi_1(x)$ and $\phi_2(x)$. The quadratic part of the action reads

$$S_2 = \int dt\, d^3x \left\{ -\frac{1}{2}(\phi_1, \phi_2) \begin{pmatrix} -\frac{\vec{\nabla}^2}{2m} + 2\lambda\phi_0^2 & \partial_t \\ -\partial_t & -\frac{\vec{\nabla}^2}{2m} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\}.$$

In momentum space, the matrix between the fields becomes

$$G^{-1}(\omega, \vec{p}) = \begin{pmatrix} \frac{\vec{p}^2}{2m} + 2\lambda\phi_0^2 & -i\omega \\ i\omega & \frac{\vec{p}^2}{2m} \end{pmatrix}.$$

Dispersion relation for quasi-particle excitations

$$\omega = \sqrt{\left(\frac{\vec{p}^2}{2m} + 2\lambda\phi_0^2\right)\frac{\vec{p}^2}{2m}}.$$

This is known as Bogoliubov dispersion relation.

Phonons and particles

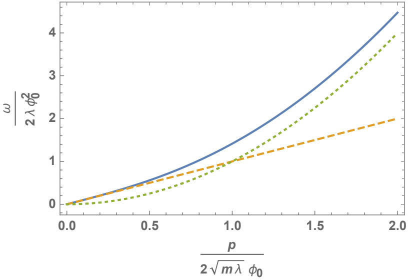
For small momenta, such that $\vec{p}^2 \ll \lambda\phi_0^2 m^2$, one finds

$$\omega \approx \sqrt{\frac{\lambda\phi_0^2}{m}}|\vec{p}|.$$

In contrast, for $\vec{p}^2 \gg \lambda\phi_0^2 m^2$ one recovers the usual dispersion relation for non-relativistic particles

$$\omega \approx \frac{\vec{p}^2}{2m}.$$

The low-momentum region describes phonons (quasi-particles of sound excitations), while the large-momentum region describes normal particles.



Landau's criterion for superfluidity

Excitations leading to friction are created for small temperatures only when

$$\epsilon(\vec{p}) + \vec{p} \cdot \vec{v} < 0.$$

Otherwise there is no viscosity and the flow is superfluid. This happens for motion below the critical velocity

$$v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{|\vec{p}|},$$

For the Bogoliubov dispersion relation the critical velocity equals the velocity of sound

$$v_c = c_s = \sqrt{\frac{\lambda \phi_0^2}{m}}.$$

For larger relative velocities, superfluidity is lost.

Additional material

Emergent global symmetries

- real scalar: $U(1) \;\rightarrow\; Q = \int d^3x\sqrt{-g}|\psi|^2$
  broken by interaction, restored by approximation method

$\rightarrow$ conservation of particle number

- complex scalar: $U(2)$, broken by interaction down to $U(1)$ with
  $Q = \int d^3x\sqrt{-g}(|\Psi_1|^2 - |\Psi_2|^2)$

$\rightarrow$ conservation of charge

Interpretation as Bogoliubov transformation

- associate non-relativistic fields with a new set of annihilation and creation operators
  $\psi = \int d^3p\ b_p e^{-ipx}$, $\psi^* = \int d^3p\ b_p^\dagger e^{ipx}$
- can express these in terms of the relativistic operators a_p, $a_p^\dagger$
- Bogoliubov transformation: $b_p = u_p a_p + v_p^* a_{-p}^\dagger$, $b_p^\dagger = u_p^* a_p^\dagger + v_p a_{-p}$,
  $|u_p|^2 - |v_p|^2 = 1$
- works for general background metric
- flat spacetime: $u_p = e^{imt},\ v_p = 0\ \rightarrow$ same vacuum