Cutting rules on a cylinder

and quantum kinetic theory for leptogenesis and dark matter models

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Outline

→ **Boltzmann equation for the early universe**

→ **Seesaw type-I reaction rates for heavy neutrinos**

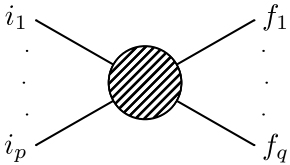
→ **Unitarity and CP asymmetries at zero temperature**

→ **Quantum thermal corrections at any perturbative order**

Boltzmann equation for the early universe

Temporal evolution of the number density of the i_1 particle species

$$\dot{n}_{i_1} + 3Hn_{i_1} = -\mathring{\gamma}_{fi} + \mathring{\gamma}_{if} + \dots \qquad (1)$$



$$\boxed{\mathring{\gamma}_{fi} = \int \prod_{\{i\}} [d\mathbf{p}_i] \mathring{f}_i(p_i) \int \prod_{\{f\}} [d\mathbf{p}_f] (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathring{M}_{fi}|^2} \qquad (2)$$

where $[d\mathbf{p}] = \dfrac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}$, in equilibrium $\mathring{f}^{\mathrm{eq}}(p) = \exp\left\{-\dfrac{E_{\mathbf{p}} - \mu}{T}\right\}$, and $|\mathring{M}_{fi}|^2$ is obtained using zero-temperature Feynman rules.

Effects of quantum statistics

$$\gamma_{fi} = \int \prod_{\{i\}} [d\mathbf{p}_i] f_i(p_i) \int \prod_{\{f\}} [d\mathbf{p}_f] \boxed{(1 \pm f_f(p_f))} (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathring{M}_{fi}|^2 \tag{3}$$

with $f^{\text{eq}}(p) = \dfrac{1}{\exp\left\{\frac{E_{\mathbf{p}} - \mu}{T}\right\} \mp 1}$.

Effects of quantum statistics

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with $f^{\mathrm{eq}}(p) = \dfrac{1}{\exp\left\{\frac{E_{\mathbf{p}} - \mu}{T}\right\} \mp 1}$.

Works fine if there are no on-shell intermediate states (to be subtracted to avoid double-counting) in $\mathring{M}_{fi}$.

$$\langle \mathbb{T}\phi(x)\phi(0)\rangle \to \int \frac{d^4p}{(2\pi)^4} e^{-\mathrm{i}p.x} \left(\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + \boxed{2\pi f(|p^0|)\delta(p^2 - m^2)} \right) \tag{4}$$

Seesaw type-I and $N_i \to lH$ decays

$$\mathcal{L} \supset -\mathcal{Y}_{\alpha i} \bar{N}_i P_L l_\alpha H + \text{H.c.} \tag{5}$$

What is the full list of processes contributing to the $n_{N_i}(t)$ evolution to the $\mathcal{O}(\mathcal{Y}^2)$ order?

Seesaw type-I and $N_i \to lH$ decays

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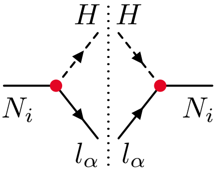
$$\mathring{\gamma}_{N\to lH} = \int [d\mathbf{p}_N]\mathring{f}_N \int [d\mathbf{p}_H][d\mathbf{p}_l](2\pi)^4\delta^{(4)}(p_N - p_l - p_H)|\mathring{M}_{N\to lH}|^2 \tag{6}$$



(a)

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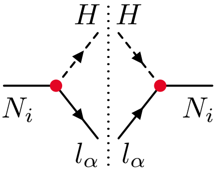
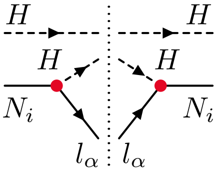


(a)



(a′)

Seesaw type-I and $N_i \to lH$ decays

$$\mathcal{L} \supset -\mathcal{Y}_{\alpha i} \bar{N}_i P_L l_\alpha H + \text{H.c.} \tag{5}$$

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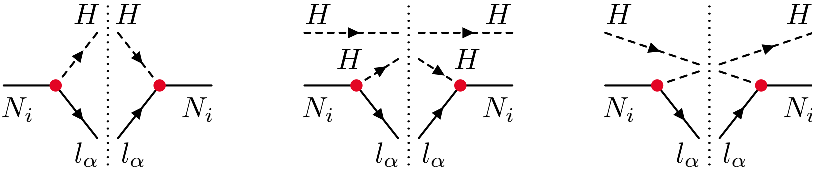
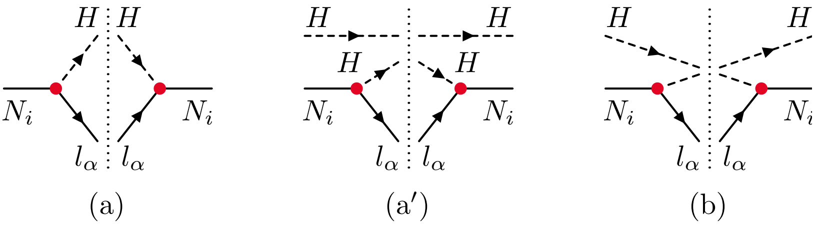


(a) (a′) (b)

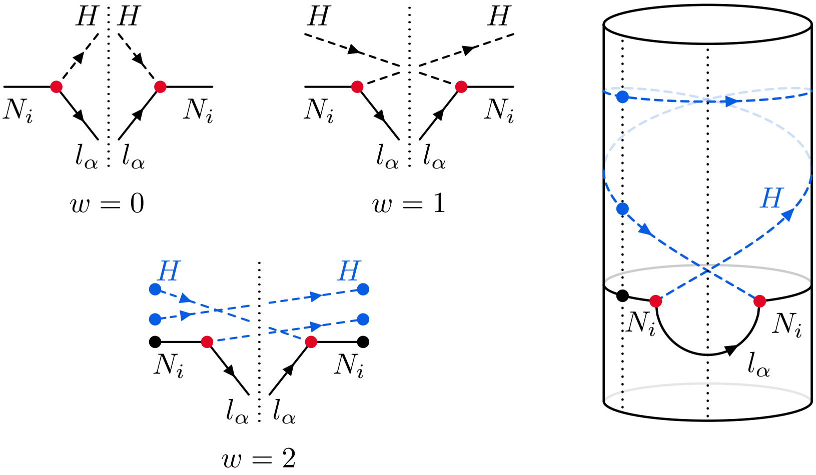
Seesaw type-I and $N_i \to lH$ decays



$$\mathring{\gamma}_{N_i \to lH} = \int [d\mathbf{p}_{N_i}] \mathring{f}_{N_i} \int [d\mathbf{p}_H][d\mathbf{p}_l](2\pi)^4 \delta^{(4)}(p_{N_i} - p_l - p_H)|\mathring{M}_{N \to lH}|^2 \tag{7a}$$

$$\mathring{\gamma}_{N_i H \to lHH} = \int [d\mathbf{p}_{N_i}][d\mathbf{p}_H] \mathring{f}_{N_i} \mathring{f}_H^{\mathrm{eq}} \int [d\mathbf{p}_l](2\pi)^4 \delta^{(4)}(p_{N_i} - p_l - p_H)|\mathring{M}_{N \to lH}|^2 \tag{7b}$$

Cylindrical diagrams and quantum statistics



$$\sum_{w=0}^{\infty}\left(\mathring{f}_H^{\mathrm{eq}}\right)^w = \frac{1}{1-\exp\{-E_H/T\}} = 1+\frac{1}{\exp\{E_H/T\}-1} = \boxed{1+f_H^{\mathrm{eq}}} \tag{8}$$

Cylindrical diagrams and quantum statistics

Similarly, for N_i and l_α

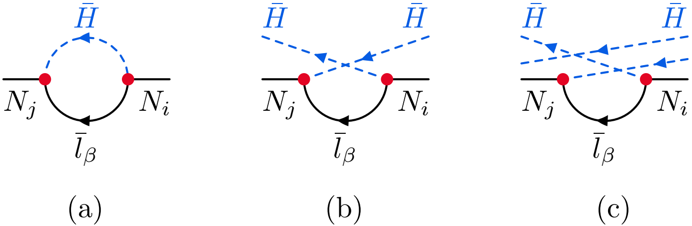
$$f_{N_i} = \sum_{w=0}^{\infty} (-1)^w \left(\mathring{f}_{N_i} \right)^{w+1} = \frac{1}{\exp\left\{ \left(E_{N_i} - \mu_{N_i} \right) / T \right\} + 1} \tag{9a}$$

$$1 - f_l^{\mathrm{eq}} = \sum_{w=0}^{\infty} (-1)^w \left(\mathring{f}_l^{\mathrm{eq}} \right)^{w} = \frac{\exp\left\{ E_l / T \right\}}{\exp\left\{ E_l / T \right\} + 1} \tag{9b}$$

such that

$$\gamma_{N_i \to lH} = \int \dots \boxed{f_{N_i} (1 - f_l^{\mathrm{eq}})(1 + f_H^{\mathrm{eq}})} \, |\mathring{M}_{N_i \to lH}|^2 . \tag{10}$$

Thermal propagators



$$\frac{\mathrm{i}}{p_{\bar{H}}^2 + \mathrm{i}\epsilon} + 2\pi \sum_{w=1}^{\infty} \mathring{f}_{\bar{H}}^{w} \theta(p_{\bar{H}}^0)\delta(p_{\bar{H}}^2) \quad \rightarrow \quad \frac{\mathrm{i}}{p_{\bar{H}}^2 + \mathrm{i}\epsilon} + \boxed{2\pi f(p_{\bar{H}}^0)\theta(p_{\bar{H}}^0)\delta(p_{\bar{H}}^2)} \tag{11}$$

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S-matrix unitarity at $T = 0$

Using $S^\dagger S = 1$ for $S_{fi} = \delta_{fi} + \mathrm{i}T_{fi} = \delta_{fi} + \mathrm{i}(2\pi)^4\delta^{(4)}(P_f - P_i)\mathring{M}_{fi}$, we obtain

$$1 - \mathrm{i}T^\dagger = (1 + \mathrm{i}T)^{-1} = 1 - \mathrm{i}T + (\mathrm{i}T)^2 - (\mathrm{i}T)^3 + \dots \tag{12}$$

such that

$$\mathrm{i}T^\dagger_{if} = \mathrm{i}T_{if} - \sum_n \mathrm{i}T_{in}\mathrm{i}T_{nf} + \sum_{n,m} \mathrm{i}T_{in}\mathrm{i}T_{nm}\mathrm{i}T_{mf} - \dots \tag{13}$$

$$\boxed{\mathring{\gamma}_{fi} = -\frac{1}{V_4}\int\prod_{\{i\}}[d\mathbf{p}_i]\mathring{f}_i(p_i)\int\prod_{\{f\}}[d\mathbf{p}_f]\left(\mathrm{i}T_{if}\mathrm{i}T_{fi} - \sum_n \mathrm{i}T_{in}\mathrm{i}T_{nf}\mathrm{i}T_{fi} + \dots\right)} \tag{14}$$

$CP(T)$ asymmetries in general

In a CPT symmetric quantum theory, the CP asymmetry is defined as $\Delta|T_{fi}|^2 = |T_{fi}|^2 - |T_{if}|^2$ and

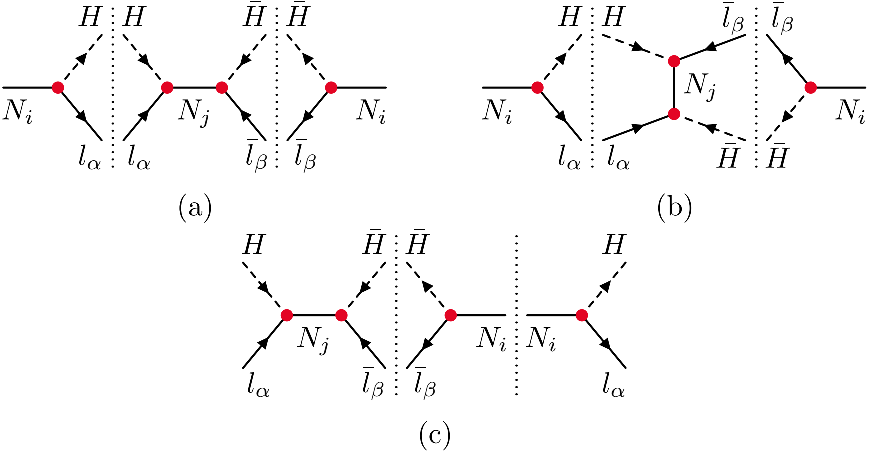
$$\begin{aligned}\Delta|T_{fi}|^2 = -\mathrm{i}T^\dagger_{if}\mathrm{i}T_{fi} + \mathrm{i}T_{if}\mathrm{i}T^\dagger_{fi} &= \sum_n (\mathrm{i}T_{in}\mathrm{i}T_{nf}\mathrm{i}T_{fi} - \mathrm{i}T_{if}\mathrm{i}T_{fn}\mathrm{i}T_{ni}) \\ &\quad - \sum_{n,m}(\mathrm{i}T_{in}\mathrm{i}T_{nm}\mathrm{i}T_{mf}\mathrm{i}T_{fi} - \mathrm{i}T_{if}\mathrm{i}T_{fm}\mathrm{i}T_{mn}\mathrm{i}T_{ni}) \\ &\quad + \ldots = -\frac{1}{V_4}(2\pi)^4\delta^{(4)}(P_f - P_i)\Delta|\mathring{M}_{fi}|^2\end{aligned} \tag{15}$$

Blažek, T., Maták, P., Phys. Rev. D103 (2021) L091302

Unitarity constraints

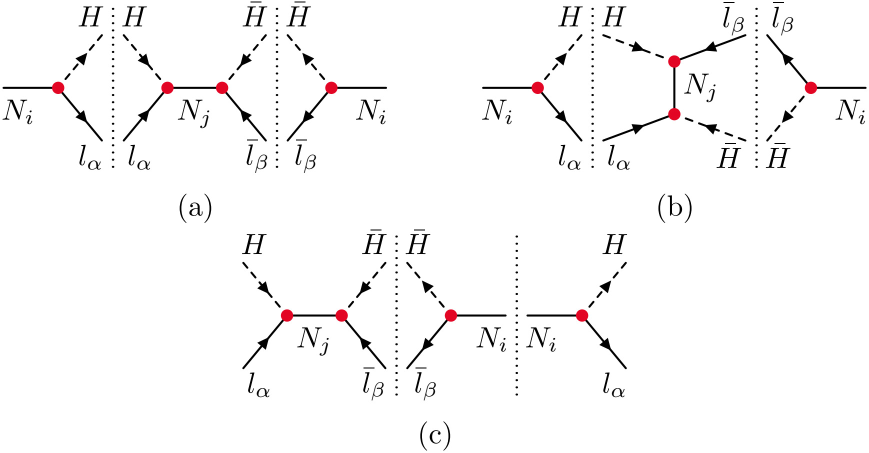
$$\boxed{\sum_f \Delta|T_{fi}|^2 = 0 \quad \rightarrow \quad \sum_f \Delta\mathring{\gamma}^{\mathrm{eq}}_{fi} = 0} \tag{16}$$

CP asymmetries at $\mathcal{O}(\mathcal{Y}^4)$



$$\Delta\gamma_{N_i\to lH} = \int \dots f_{N_i}\left(1 - f_l^{\rm eq}\right)\left(1 + f_H^{\rm eq}\right)\left(1 - f_{\bar{l}}^{\rm eq}\right)\left(1 + f_{\bar{H}}^{\rm eq}\right)\Delta|\mathring{M}_{N_i\to lH}|^2 \tag{17}$$

CP asymmetries at $\mathcal{O}(\mathcal{Y}^4)$



$$\dot{n}_L + 3Hn_L = \int \ldots \delta f_{N_i} \left(1 - f_l^{\rm eq}\right) \left(1 + f_H^{\rm eq}\right) \boxed{\left(1 - f_{\bar{l}}^{\rm eq} + f_{\bar{H}}^{\rm eq}\right)} \Delta|\mathring{M}_{N_i \to lH}|^2 + \ldots \qquad (18)$$

Garny, M. *et al.*, Phys. Rev. D80 (2009) 125027

General one-particle densities

The hermiticity and positive definiteness of $\hat{\rho}$ allows us to write

$$\hat{\rho} = \frac{1}{\mathcal{Z}} \exp\left\{-\hat{\mathcal{F}}\right\}, \mathcal{Z} = \mathrm{Tr}\exp\left\{-\hat{\mathcal{F}}\right\}. \tag{19}$$

Assuming molecular chaos, we set

$$\hat{\mathcal{F}} = \sum_p \mathcal{F}_p a_p^\dagger a_p \tag{20}$$

Wagner, M., Phys. Rev. B44 (1991) 6104–6117

$$\mathcal{Z} = \sum_{\{i\}} \exp\{-\mathcal{F}_1 i_1 - \mathcal{F}_2 i_2 - \ldots\} = \prod_p \mathcal{Z}_p \quad \text{kde} \quad \mathcal{Z}_p = \frac{\exp\{\mathcal{F}_p\}}{\exp\{\mathcal{F}_p\} - 1}, \tag{21}$$

$$f_p = \mathrm{Tr}\left[\hat{\rho} a_p^\dagger a_p\right] = \frac{1}{\exp\{\mathcal{F}_p\} - 1} \quad \Rightarrow \quad \boxed{\mathring{f}_p \stackrel{\text{def.}}{=} \exp\left\{-\mathcal{F}_p\right\}}$$

General one-particle densities

$$\hat{\rho}' = S\hat{\rho}S^\dagger \quad \Rightarrow \quad \hat{\rho}' - \hat{\rho} = T\hat{\rho}T^\dagger - \frac{1}{2}TT^\dagger\hat{\rho} - \frac{1}{2}\hat{\rho}TT^\dagger + \dots \tag{22}$$

McKellar, B.H.J., Thomson, M.J., Phys. Rev. D49 (1994) 2710–2728

Tracing with $a_p^\dagger a_p$ over $|i_1, i_2, \dots\rangle$ we get

$$\begin{aligned} f_p' - f_p &= \mathrm{Tr}\Big[a_p^\dagger a_p\Big(T\hat{\rho}T^\dagger - \hat{\rho}TT^\dagger\Big)\Big] \\ &= \sum_{k=1}^{\infty}(-1)^k \mathrm{Tr}\Big[a_p^\dagger a_p\Big(\mathrm{i}T\hat{\rho}(\mathrm{i}T)^k - \hat{\rho}(\mathrm{i}T)^{k+1}\Big)\Big] \\ &= \boxed{\frac{1}{\mathcal{Z}}\sum_{k=1}^{\infty}(-1)^k \sum_{\{i\}}\sum_{\{n\}}\left(n_p - i_p\right)\mathring{f}_1^{i_1}\mathring{f}_2^{i_2}\dots(\mathrm{i}T)^k_{in}\mathrm{i}T_{ni}} \end{aligned} \tag{23}$$

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Conclusions

- $\leftarrow$ **A diagrammatic concept connecting the classical Boltzmann equation and quantum kinetic theory has been introduced.**

- $\leftarrow$ **Statistical factors due to on-shell intermediate states have been formally represented by the cuts of forward diagrams with multiple spectator lines.**

- $\leftarrow$ **Results are in agreement with the direct closed-time-path derivation of non-equilibrium quantum field theory.**

- $\leftarrow$ **Unitarity and CPT constraints to the reaction rate asymmetries can be derived in our approach to kinetic theory in the same way as in $T = 0$ calculations.**

Thank you for your attention!