RESONANT LEPTOGENESIS IN B-L EXTENDED STANDARD MODEL AT TEV SCALE

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CTU, IEAP

CONTENTS

RESONANT LEPTOGENESIS FLAVOR SYMMETRY

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RESONANT LEPTOGENESIS

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BARYON ASYMMETRY

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There are baryon asymmetry in universe.

5.8
$$10^{-10} < \frac{n_b - n_{\bar{b}}}{n_{\gamma}} < 6.5 \ 10^{-10}$$

The origin of baryon asymmetry is mystery.

SAKHAROV'S CONDITION

THE BARYON ASYMMETRY CAN BE DYNAMICALLY GENERATED ('BARYOGENESIS') PROVIDED THAT

1. B VIOLATION

2. C AND CP VIOLATION

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3. OUT OF THERMAL EQUILIBRIUM

BARYOGENESIS VIA LEPTOGENESIS

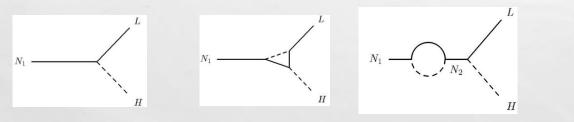
SM + right-handed Majorana neutrinos

- **o B** violation
 - L violation by majorana neutrinos decay
 - **Sphaleron process**
- C and CP violation
 - **Neutrino mixing matrix (complex)**
- o Out of thermal equilibrium

Out of equilibrium decay of right-handed neutrino

BARYOGENESIS VIA LEPTOGENESIS

• L ASYMMETRY IS GENERATED DUE TO CP ASYMMETRY THAT ARISES THROUGH INTERFERENCE OF TREE LEVEL AND ONE-LOOP DIAGRAMS.



SPHALERON PROCESS CONVERTS LASYMMETRY INTO B ASYMMETRY.

CP ASYMMETRY

 $\epsilon_{1} \equiv \frac{\sum_{j} \left[\Gamma\left(N_{1} \rightarrow \ell_{j}h\right) - \Gamma\left(N_{1} \rightarrow \ell_{j}^{C}h^{*}\right) \right]}{\sum_{j} \left[\Gamma\left(N_{1} \rightarrow \ell_{j}h\right) + \Gamma\left(N_{1} \rightarrow \ell_{j}^{C}h^{*}\right) \right]}$ $= -\sum_{j=2,3} \frac{M_{1}}{M_{j}} \frac{\Gamma_{j}}{M_{j}} \left(\frac{V_{j}}{2} + S_{j}\right) \frac{\operatorname{Im}\left[(Y_{D}^{2})_{1j}^{2}\right]}{\left|Y_{D}^{2}\right|_{11}} \left|Y_{D}^{2}\right|_{jj}}$

Right-handed Majorana neutrino can decay into lepton and anti-lepton.

Vertex contribution

Self-energy contribution

 $S_{j} = \frac{M_{j}^{2} \Delta M_{1j}^{2}}{\left(\Delta M_{1j}^{2}\right)^{2} + M_{1}^{2} \Gamma_{j}^{2}}$

 $\frac{\Gamma_j}{M_j} = \cdot$

 $\Delta M_{1i}^2 = M_i^2 - M_1^2$

$$V_{j} = 2 \frac{M_{j}^{2}}{M_{1}^{2}} \left[\left(1 + \frac{M_{j}^{2}}{M_{1}^{2}} \right) \log \left(1 + \frac{M_{1}^{2}}{M_{j}^{2}} \right) - 1 \right]$$

HIERARCHICAL CASE

If right-handed neutrinos have a hierarchical mass spectrum ($M_{2,3} \gg M_1$), we can write a CP asymmetry parameter as

$$\epsilon_1 \sim \frac{3}{16\pi} \frac{m_{\nu} M_1}{v^2} \sin \delta \sim 10^{-6} \frac{m_{\nu}}{0.05 eV} \frac{M_1}{10^{10} GeV} \sin \delta$$

The present Baryon asymmetry

 $Y_B \sim \kappa \frac{\epsilon_1}{g_*} \sim 10^{-10}$ $g_* = \mathcal{O}(100)$ is the number of relativistic degrees of freedom. In the SM $\kappa \sim 2 \times 10^{-2} \left(\frac{0.05 eV}{m_{\nu}}\right)^{1.1}$ The efficiency factor (due to wash out)

MAJORANA MASS BOUND

$$Y_B \sim 10^{-10} \left(\frac{0.05 eV}{m_{\nu}}\right)^{0.1} \frac{M_1}{10^{10} GeV} \sin \delta \sim 10^{-10}$$

 $m_{\nu} \sim \sqrt{m_{atm}^2} \sim 0.05 eV$

The lightest Majorana mass is heavier than $10^{10}GeV$.

RESONANT LEPTOGENESIS

• THE MAJORANA MASSES IS HEAVIER THAN $10^{10}GeV$, if the spectrum of majorana masses has hierarchy.

• IF THE MAJORANA MASS OF RIGHT-HANDED NEUTRINO IS SMALLER THAN A FEW TEV, GENERAL LEPTOGENESIS CAN NOT WORK.



RESONANT-LEPTOGENESIS

 ϵ_1 can be even O(1).

If two right-handed neutrinos have mass differences comparable to their decay widths $(M_2^2 - M_1^2 \sim M_1 \Gamma_2)$, self-energy correction dominate.

$$\epsilon_{1} \sim \frac{Im[Y^{2}]_{12}^{2}}{|Y^{2}|_{11}|Y^{2}|_{22}} \frac{M_{1}\Gamma_{2}\left(M_{2}^{2}-M_{1}^{2}\right)}{\left(M_{2}^{2}-M_{1}^{2}\right)^{2}+M_{1}^{2}\Gamma_{2}^{2}} \sim \frac{1}{2} \frac{Im[Y^{2}]_{12}^{2}}{|Y^{2}|_{11}|Y^{2}|_{22}}$$

 $\frac{M_2^2 - M_1^2}{M_1^2} \sim 10^{-12}, \qquad M_1 \sim TeV$

DEGENERATE MASSES

 $M_1 = M_2$ @ High scale

RG evolution



 $M_2^2 - M_1^2 \propto Y_{\nu}^{\dagger} Y_{\nu} (y v_{B-L})^2$ \$\propto \Gamma M\$

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Flavor symmetry S₄ Modular symmetry

FLAVOR SYMMETRY

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MODEL

3 Right-Handed Neutrinos \Rightarrow Resonant Leptogenesis
+
Dark Matter candidateU(1) B-L Gauge Symmetry \Rightarrow Thermal Production of NeutrinosS4 Modular Symmetry \Rightarrow Resonant Leptogenesis



| | Fermions | | | | Bosons | | | |
|--------------|---|-----------|---|-----------|-----------------------------------|----------------|---------------|--------|
| | $\bar{L}_L \equiv (\bar{L}_{L_e}, \bar{L}_{L_{\mu}}, \bar{L}_{L_{\tau}})^T$ | e_{R_e} | $\ell_R \equiv (e_{R_\mu}, e_{R_\tau})^T$ | N_{R_1} | $N_R \equiv (N_{R_2}, N_{R_3})^T$ | H_u | H_d | ϕ |
| $SU(2)_L$ | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| $U(1)_Y$ | $\frac{1}{2}$ | -1 | -1 | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| $U(1)_{B-L}$ | 1 | -1 | -1 | -1 | -1 | 0 | 0 | 2 |
| S_4 | 3′ | 1 | 2 | 1 | 2 | 1 | 1 | 1 |
| -k | -1 | -1 | -1 | -3 | 0 | -3 | 0 | 0 |

Yukawa

$$\mathcal{L}_{Y} = \alpha \bar{L}_{L} Y_{\mathbf{3}'}^{(2)} e_{R_{e}} H_{d} + \beta \bar{L}_{L} Y_{\mathbf{3}'}^{(2)} \ell_{R} H_{d} + \bar{L}_{L} \left(\gamma_{1} Y_{\mathbf{3}}^{(4)} + \gamma_{2} Y_{\mathbf{3}'}^{(4)} \right) N_{R} H_{u} + \delta \bar{N}_{R_{1}}^{c} Y_{\mathbf{1}}^{(6)} N_{R_{1}} \phi + y \bar{N}_{R}^{c} N_{R} \phi$$

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MODULAR SYMMETRY

$$\tau \longrightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d}$$
, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, $\operatorname{Im}[\tau] > 0$

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Generators

$$S: \tau \longrightarrow -\frac{1}{\tau}, \qquad T: \tau \longrightarrow \tau + 1 \qquad S^2 = I, (ST)^3 = I$$

Infinite normal group

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \qquad \qquad T^{N} = I$$

$$\Gamma(4)\approx S_4$$

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MODULAR SYMMETRY

Holomorphic function (Yukawa couplings)

$$\begin{split} f(\gamma\tau) &= (c\tau + d)^k f(\tau) \ , \ \gamma \in \Gamma(N) \\ a &= d = -1, b = c = 0 \\ f(\tau) &= (-1)^k f(\tau) \\ \textbf{Odd modular weight} \Rightarrow \textbf{Vanish} \end{split} \begin{array}{ll} \hline \textbf{Modular Weight} & \textbf{Representation} \\ \textbf{2} & \textbf{2}, \textbf{3}' \\ \textbf{4} & \textbf{1}, \textbf{2}, \textbf{3}, \textbf{3}' \\ \textbf{6} & \textbf{1}, \textbf{1}', \textbf{2}, \textbf{3}, \textbf{3}' \\ \end{array} \end{split}$$

 $\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)}$, $\rho^{(I)}(\gamma)$: Unitary representation matrix of S_4

MODEL

Yukawa

$$\mathcal{L}_{Y} = \alpha \bar{L}_{L} Y_{\mathbf{3}'}^{(2)} e_{R_{e}} H_{d} + \beta \bar{L}_{L} Y_{\mathbf{3}'}^{(2)} \ell_{R} H_{d} + \bar{L}_{L} \left(\gamma_{1} Y_{\mathbf{3}}^{(4)} + \gamma_{2} Y_{\mathbf{3}'}^{(4)} \right) N_{R} H_{u} + \delta \bar{N}_{R_{1}}^{c} Y_{\mathbf{1}}^{(6)} N_{R_{1}} \phi + y \bar{N}_{R}^{c} N_{R} \phi$$

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Vanishing terms

| $ar{L}_L N_{R_1} H_u$ $ar{N}_R^c N_{R_1} \phi$ | Yukawa coupling with odd modular weight |
|---|---|
| N_{R_1} is stable | Dark Matter Candidate |

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| | Fermions | | | | Bosons | | | |
|--------------|---|-----------|---|-----------|-----------------------------------|----------------|---------------|--------|
| | $\bar{L}_L \equiv (\bar{L}_{L_e}, \bar{L}_{L_{\mu}}, \bar{L}_{L_{\tau}})^T$ | e_{R_e} | $\ell_R \equiv (e_{R_{\mu}}, e_{R_{\tau}})^T$ | N_{R_1} | $N_R \equiv (N_{R_2}, N_{R_3})^T$ | H_u | H_d | ϕ |
| $SU(2)_L$ | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| $U(1)_Y$ | $\frac{1}{2}$ | -1 | -1 | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| $U(1)_{B-L}$ | 1 | -1 | -1 | -1 | -1 | 0 | 0 | 2 |
| S_4 | 3′ | 1 | 2 | 1 | 2 | 1 | 1 | 1 |
| -k | -1 | -1 | -1 | -3 | 0 | -3 | 0 | 0 |

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MODEL

Boundary conditions

$$Y_{M}(m_{0}) = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \qquad m_{0}: \text{Breaking scale of the flavor symmetry} \\ Y_{\nu}(m_{0}) = \begin{pmatrix} \gamma_{2}y'_{1} & -\gamma_{1}y_{1} \\ \frac{\sqrt{3}}{2}\gamma_{1}y_{2} - \frac{1}{2}\gamma_{2}y'_{3} & \frac{1}{2}\gamma_{1}y_{3} + \frac{\sqrt{3}}{2}\gamma_{2}y'_{2} \\ \frac{\sqrt{3}}{2}\gamma_{1}y_{3} - \frac{1}{2}\gamma_{2}y'_{2} & \frac{1}{2}\gamma_{1}y_{2} + \frac{\sqrt{3}}{2}\gamma_{2}y'_{3} \end{pmatrix} \qquad Y_{3}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} y_{1$$

$$Y_{\nu}(m_{EW}) \sim U_{MNS}^{*} \begin{pmatrix} 0 & 0\\ \sqrt{m_{1}} & 0\\ 0 & \sqrt{m_{2}} \end{pmatrix} O \sqrt{Y_{M}(m_{EW})} \frac{\sqrt{v_{B-L}}}{v} \qquad O = \begin{pmatrix} \cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha \end{pmatrix}, \alpha \in \mathbb{C}$$

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 y'_1

 y'_2

 y'_3

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RESULT

parameters

Guage

 $v_{B-L} = 17 \ TeV$ $g_{B-L} = 0.20$ **Modular** $\tau = 0.25 + 2.5 \ i$

Yukawa

 $\alpha = i$ y = 0.5

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predictions Baryon number

 $\frac{n_b - n_{\bar{b}}}{n_{\gamma}} = 5.93 \ 10^{-10}$

Active neutrino

$$\begin{split} m_1 &= 8.46 \; 10^{-3} \; eV \\ m_2 &= 4.93 \; 10^{-2} \; eV \end{split}$$

Majorana neutrino

 $M_{1} = 7.79 \ TeV$ $\frac{M_{2}^{2} - M_{1}^{2}}{M_{1}^{2}} = 9.92 \ 10^{-12}$ **Gauge boson** $M_{z'} = 6.90 \ TeV$

Observed values

 $5.8 \ 10^{-10} < \frac{n_b - n_{\overline{b}}}{n_{\gamma}} < 6.5 \ 10^{-10}$ (95 % CL)

8.26 $10^{-3} eV < m_1 < 8.97 \ 10^{-3} eV$ (3 d) 4.93 $10^{-2} eV < m_2 < 5.10 \ 10^{-2} eV$ (3 d)

CONCLUSIONS

• THE RESONANT LEPTOGENESIS SCENARIO IN THE TEV SCALE B-L MODEL

• S₄ MODULAR SYMMETRY NATURALLY REALIZES DEGENERATE MASSES

- RG EVOLUTIONS GENERATE A SPLITTING OF THE MAJORANA MASSES
- THE CP ASYMMETRY PARAMETERS ARE AUTOMATICALLY ENHANCED BY RESONANCE
- THE RIGHT AMOUNT OF BARYON NUMBER CAN BE OBTAINED

FUTURE WORK

• SEARCH WIDE REGION

• QUARK AND CHARGED LEPTON SECTOR

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Representations : 1, 1', 2, 3, 3'

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Tensor Products

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| | $2 \otimes 2 = 1 \oplus 1' \oplus 2$ |
|---------------------|---|
| $1' \otimes 1' = 1$ | $2 \otimes 3 = 3 \oplus 3'$ |
| $1' \otimes 2 = 2$ | $2 \otimes 3' = 3 \oplus 3'$ |
| $1' \otimes 3 = 3'$ | $3 \otimes 3 = 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3'$ |
| $1' \otimes 3' = 3$ | $3 \otimes 3' = 1' \oplus 2 \oplus 3 \oplus 3'$ |

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YUKAWA COUPLINGS

$$Y_{2}^{(2)} = i \begin{pmatrix} Y^{(4)}(1, 1, -1/2, -1/2, -1/2, -1/2 | \tau) \\ Y^{(4)}(0, 0, \sqrt{3}/2, -\sqrt{3}/2, \sqrt{3}/2, -\sqrt{3}/2 | \tau) \end{pmatrix} \qquad Y_{3\prime}^{(2)} = i \begin{pmatrix} Y^{(4)}(1, -1, 0, 0, 0, 0 | \tau) \\ Y^{(4)}(0, 0, -1/\sqrt{2}, i/\sqrt{2}, 1/\sqrt{2}, -i/\sqrt{2} | \tau) \\ Y^{(4)}(0, 0, -1/\sqrt{2}, -i/\sqrt{2}, 1/\sqrt{2}, i/\sqrt{2} | \tau) \end{pmatrix}$$

$$Y^{(4)}(a_1, \dots, a_6 | \tau) = \frac{d}{d\tau} \left[a_1 \log \eta \left(\tau + \frac{1}{2} \right) + a_2 \log \eta \left(4\tau \right) + a_3 \log \eta \left(\frac{\tau}{4} \right) \right. \\ \left. + a_4 \log \eta \left(\frac{\tau + 1}{4} \right) + a_5 \log \eta \left(\frac{\tau + 2}{4} \right) + a_6 \log \eta \left(\frac{\tau + 3}{4} \right) \right]$$

Dedekind eta function

 $Y_3^{(2)} = 0$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) , \qquad q = e^{2\pi i}$$

Multiplets of higher weight can be obtained from those of lowest weight via tensor products.

MODULAR FORM

$$\begin{split} Y^{(4,2)}_{\mathbf{2}}(\tau) &= -3\pi \, \left(\begin{array}{c} b^{(4)}_1(\tau)/8 + 3\, b^{(4)}_5(\tau) \\ -\sqrt{3}\, b^{(4)}_3(\tau) \end{array} \right) \,, \\ Y^{(4,2)}_{\mathbf{3'}}(\tau) &= -\pi \, \left(\begin{array}{c} -b^{(4)}_1(\tau)/4 + 2\, b^{(4)}_5(\tau) \\ \sqrt{2}\, b^{(4)}_2(\tau) \\ 4\sqrt{2}\, b^{(4)}_4(\tau) \end{array} \right) \,. \end{split}$$

$$\begin{split} b_1^{(4)} &= 1 + 24q_4^8 + 24q_4^{16} + 96q_4^{24} + 24q_4^{32} + 144q_4^{40} + \dots, \\ b_2^{(4)} &= q_4 + 6q_5^5 + 13q_4^9 + 14q_4^{13} + 18q_4^{17} + 32q_4^{21} + 31q_4^{25} + 30q_4^{29} + 48q_4^{33} + 38q_4^{37} + \dots, \\ b_3^{(4)} &= q_4^2 + 4q_4^6 + 6q_4^{10} + 8q_4^{14} + 13q_4^{18} + 12q_4^{22} + 14q_4^{26} + 24q_4^{30} + 18q_4^{34} + 20q_4^{38} + \dots, \\ b_4^{(4)} &= q_4^3 + 2q_4^7 + 3q_4^{11} + 6q_4^{15} + 5q_4^{19} + 6q_4^{23} + 10q_4^{27} + 8q_4^{31} + 12q_4^{35} + 14q_4^{39} + \dots, \\ b_5^{(4)} &= q_4^4 + 4q_4^{12} + 6q_4^{20} + 8q_4^{28} + 13q_4^{36} + \dots, \end{split}$$

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 $q_4 \equiv e^{2\pi i \, \tau/4} = e^{\pi i \, \tau/2}$