Department of Physics. University of La Plata. Argentina

Multipole decomposition of the rate of $\mu^- \to e^-$ conversion in ^{208}Pb

O.Civitarese and T.Tarutina

May 21, 2019

Content of the talk

- Motivations
- Formalism
- Results
- Conclusions

Motivations

- Write down a general Lagrangian to describe LFV processes
- Disentangle nuclear structure and particle physics components of the problem
- Present a formalism friendly to model building of LFV mechanisms.
- Give an example of the method (the case of lead 208)

$$\mu^- + (A, Z) \longrightarrow e^- + (A, Z)^*$$

$$|p_e| = m_\mu - \epsilon_b - (E_f - E_i),$$

The probability of the muon-to-electron conversion process reads

$$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \int d\hat{\mathbf{p}}_e \left(\frac{p_e}{m_\mu} \right)^2 |\langle f | T_{nucl} | i, \mu \rangle|^2 \; T_{lep},$$

Formalism
The Lagrangian

The most general effective lepton-flavor violation (LFV) Lagrangian in the non-photonic sector and at the quark-lepton level can be written in the form:

$$
\begin{aligned}
\mathcal{L}_{np} = & -\frac{G_F}{\sqrt{2}} \sum_{q=u,d,s,..} [(g_{LS}(q)\overline{e}_L\mu_R + g_{RS}(q)\overline{e}_R\mu_L)\overline{q}q \\
& +(g_{LP}(q)\overline{e}_L\gamma_5\mu_R + g_{RP}(q)\overline{e}_R\gamma_5\mu_L)\overline{q}\gamma_5 q \\
& +(g_{LV}(q)\overline{e}_L\gamma^\mu\mu_R + g_{RV}(q)\overline{e}_R\gamma^\mu\mu_L)\overline{q}\gamma_\mu q \\
& +(g_{LA}(q)\overline{e}_L\gamma^\mu\gamma_5\mu_L + g_{RA}(q)\overline{e}_R\gamma^\mu\gamma_5\mu_R)\overline{q}\gamma_\mu\gamma_5 q \\
& +\frac{1}{2}(g_{LT}(q)\overline{e}_L\sigma^{\mu\nu}\mu_R + g_{RT}(q)\overline{e}_R\sigma^{\mu\nu}\mu_L)\overline{q}\sigma_{\mu\nu}q] \\
& =\mathcal{L}_S + \mathcal{L}_P + \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_T,
\end{aligned}
$$

G_F is the Fermi constant, $g_{XK}(q)$ are dimensionless coupling constants at quark level (q), $X = \{L, R\}$ and $K = \{S, P, V, A, T\}$ (scalar, pseudo-scalar, vector, axial-vector, and tensor terms).

Formalism
Lagrangian at the nucleon level

The Lagrangian can be rewritten using the isospin formalism at the nucleonic level:

$$\begin{aligned}\mathcal{L}_{np} = -\frac{G_F}{\sqrt{2}}[&\overline{e}_L\mu_R\overline{\psi}(g^{(0)}_{LS}+g^{(1)}_{LS}\tau_3)\psi+\overline{e}_R\mu_L\overline{\psi}(g^{(0)}_{RS}+g^{(1)}_{RS}\tau_3)\psi\\ &+\overline{e}_L\gamma^5\mu_R\overline{\psi}\gamma_5(g^{(0)}_{LP}+g^{(1)}_{LP}\tau_3)\psi+\overline{e}_R\gamma^5\mu_L\overline{\psi}\gamma_5(g^{(0)}_{RP}+g^{(1)}_{RP}\tau_3)\psi\\ &+\overline{e}_L\gamma^\mu\mu_L\overline{\psi}\gamma_\mu(g^{(0)}_{LV}+g^{(1)}_{LV}\tau_3)\psi+\overline{e}_R\gamma^\mu\mu_R\overline{\psi}\gamma_\mu(g^{(0)}_{RV}+g^{(1)}_{RV}\tau_3)\psi\\ &+\overline{e}_L\gamma^\mu\gamma^5\mu_L\overline{\psi}\gamma_\mu\gamma_5(g^{(0)}_{LA}+g^{(1)}_{LA}\tau_3)\psi+\overline{e}_R\gamma^\mu\gamma^5\mu_R\overline{\psi}\gamma_\mu\gamma_5(g^{(0)}_{RA}+g^{(1)}_{RA}\tau_3)\psi\\ &+\frac{1}{2}\left(\overline{e}_L\sigma^{\mu\nu}\mu_L\overline{\psi}\sigma_{\mu\nu}(g^{(0)}_{LT}+g^{(1)}_{LT}\tau_3)\psi+\overline{e}_R\sigma^{\mu\nu}\mu_R\overline{\psi}\sigma_{\mu\nu}(g^{(0)}_{RT}+g^{(1)}_{RT}\tau_3)\psi\right)],\end{aligned}$$

where $\psi = (p, n)^T$, and $g^{(0)}_{XK}$ and $g^{(1)}_{XK}$ are isoscalar and isovector coupling constants.

The coupling constants are connected with coupling constants at the quark level through:

$$g_{XK}^{(0)} = \frac{1}{2}\sum_{q=u,d,s}(g_{XK}(q)G_K^{(q,p)} + g_{XK}(q)G_K^{(q,n)})$$

$$g_{XK}^{(1)} = \frac{1}{2}\sum_{q=u,d,s}(g_{XK}(q)G_K^{(q,p)} - g_{XK}(q)G_K^{(q,n)}),$$

where $G_K^{(q,p)}$ are form-factors obeying the relations

$$G_K^{(u,p)} = G_K^{(d,n)},\ G_K^{(d,p)} = G_K^{(u,n)},\ G_K^{(s,p)} = G_K^{(s,n)},$$

where u, d, and s are quark indices and p and n stand for protons and neutrons.

Table: Form factors at the quark level

$G_V^{(u,p)} = 2$	$G_V^{(d,p)} = 1$	$G_V^{(s,p)} = 0$
$G_A^{(u,p)} = 0.78$	$G_A^{(d,p)} = -0.47$	$G_A^{(s,p)} = -0.19$
$G_S^{(u,p)} = 5.1$	$G_S^{(d,p)} = 4.3$	$G_S^{(s,p)} = 2.5$
$G_P^{(u,p)} = 103$	$G_P^{(d,p)} = 100$	$G_P^{(s,p)} = 3.3$

The coupling constants for neutron $g_{XK}(n)$ and proton $g_{XK}(p)$ states are related to $g_{XK}^{(0)}$ and $g_{XK}^{(1)}$ by:

$$
\begin{aligned}
g_{XK}(n) &= g_{XK}^{(0)} - g_{XK}^{(1)} \\
g_{XK}(p) &= g_{XK}^{(0)} + g_{XK}^{(1)}.
\end{aligned}
$$

Operators

To obtain the expression for $M_{\rm Nucl}$ the effective interaction is expanded in powers of the inverse nucleon mass M_N^{-1} and in the limit of small momenta, $p_N/M_N \ll 1$, where p_N is the momentum of the nucleon, one obtains:

$$M^2_{\rm Nucl}(J_f^\pi) = M^2(\langle \mathbf{1} \rangle) + M^2(\langle \sigma \rangle) + M^2(\Lambda),$$

$M^2(\langle \mathbf{1} \rangle)$, $M^2(\langle \sigma \rangle)$ and $M^2(\Lambda)$ are the spin-independent, spin-dependent and tensor parts of the squared matrix element.

$$M^2(\langle \mathbf{1} \rangle) = m_e c_1 + E_e c_2 - \frac{1}{2M_N} p_e^2 c_3,$$

$$\begin{aligned} c_1 &= \sum_{qq'=(p,n)} \{2g_{LS}(q)g_{RS}(q') + 2g_{LS}(q)g_{RV}(q') + 2g_{RV}(q)g_{LV}(q') \\ &+ g_{RS}(q)g_{LV}(q') + g_{RS}(q)g_{RV}(q')\}\mathcal{M}(qq', I\,\gamma = 0\ J) \end{aligned}$$

$$\begin{aligned} c_2 &= \sum \{2g_{LS}(q)g_{LV}(q') + 2g_{RS}(q)g_{RV}(q') + 2g_{RV}(q)g_{RV}(q') \\ &+ g_{LS}(q)g_{LS}(q') + g_{RS}(q)g_{RS}(q')\}\mathcal{M}(qq', I\,\gamma = 0\ J) \end{aligned}$$

$$\begin{aligned} c_3 &= \sum \{g_{LS}(q)g_{LV}(q') + g_{RS}(q)g_{RV}(q') - g_{LS}(q)g_{LT}(q') \\ &- g_{RS}(q)g_{RT}(q') + g_{RV}(q)g_{RV}(q') + g_{LV}(q)g_{LV}(q') \\ &- g_{RV}(q)g_{RT}(q') - g_{LV}(q)g_{LV}(q')\}\mathcal{M}(qq', I\,\gamma = 0\ J), \end{aligned}$$

Operators
Spin dependent

$$M^2(\langle\sigma\rangle)) = m_e c_1' + E_e c_2' - \frac{1}{2M_N} p_e^2 c_3',$$

$$c_1' = \sum\{2g_{RA}(q)g_{LA}(q') + 2g_{RA}(q)g_{LT}(q') + 2g_{LA}(q)g_{RT}(q') + 2g_{LT}(q)g_{RT}(q')\}\mathcal{M}(qq', I\,\gamma = 1\ J)$$

$$c_2' = \sum\{2g_{RA}(q)g_{RT}(q') + 2g_{LA}(q)g_{LT}(q') + g_{RA}(q)g_{RA}(q') + g_{LA}(q)g_{LA}(q') + g_{LT}(q)g_{LT}(q') + g_{RT}(q)g_{RT}(q')\}\mathcal{M}(qq', I\,\gamma = 1\ J)$$

$$c_3' = \sum\{2(-g_{LP}(q)g_{LA}(q') - g_{RP}(q)g_{RA}(q') - g_{LP}(q)g_{LT}(q') - g_{RP}(q)g_{RT}(q') + g_{RA}(q)g_{RA}(q') + g_{LA}(q)g_{LA}(q') - g_{RA}(q)g_{RT}(q') - g_{LA}(q)g_{LT}(q'))\}\mathcal{M}(qq', I\,\gamma = 1\ J),$$

Operators
Tensor terms

$$M^2(\Lambda) = -\frac{1}{2M_N} p_e^2 \sqrt{\frac{2}{3}} c_3'',$$

$$c_3'' = \sum_{qq'=(p,n)} \{(-2g_{LP}(q)g_{LA}(q') - 2g_{RP}(q)g_{RA}(q') - 2g_{LP}(q)g_{LT}(q') \\ -2g_{RP}(q)g_{RT}(q') + g_{RA}(q)g_{RA}(q') + g_{LA}(q)g_{LA}(q') \\ -g_{LT}(q)g_{LT}(q') - g_{RT}(q)g_{RT}(q'))\}\mathcal{M}(qq'),$$

Nuclear matrix elements

$$\mathcal{M}(qq', l\ \gamma = 0\ J) = \sum_{k,k',\rho,\rho'} C^{(k)}(\rho) C^{(k')*}(\rho') \langle \rho, k || T^{(l,0)J} || 0^+ \rangle \langle \rho', k' || T^{(l,0)J} || 0^+ \rangle^* .$$

$$\mathcal{M}(qq', l\ \gamma = 1\ J) = \sum_{k,k',\rho,\rho'} C^{(k)}(\rho) C^{(k')*}(\rho') \langle \rho, k || T^{(l,1)J} || 0^+ \rangle \langle \rho', k' || T^{(l,1)J} || 0^+ \rangle^*$$

$$\mathcal{M}(qq')_{tensor} = \sum_{kk',\rho,\rho'} C^{(k)}(\rho) C^{(k')*}(\rho') \left(\frac{5}{6}\right)^2 \frac{4\pi}{2J_i + 1} \sum_{l,l',K} (-1)^{l/2 - l'/2 + K} \sqrt{(2l+1)(2l'+1)} \begin{pmatrix} l & l' & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 1 & 1 & 2 \\ l' & l & K \end{Bmatrix} \langle \rho, k || i^l j_l(p_e r) g(r) [Y_l(\hat{\mathbf{r}}) \times \sigma]^{(K)} || 0^+ \rangle \langle \rho', k' || i^{l'} j_l'(p_e r) g(r) [Y_{l'}(\hat{\mathbf{r}}) \times \sigma]^{(K)} || 0^+ \rangle^* . \quad (1)$$

Radial dependence of the operators
muon wave functions

$$T^{(l,0)JM} = \delta_{lJ}\sqrt{4\pi}\, i^l j_l(p_e r) g(r) Y_{lM}(\hat{r}),$$
$$T^{(l,1)JM} = \sqrt{4\pi}\, i^l j_l(p_e r) g(r) [Y_l(\hat{r}) \times \sigma]^{(JM)}.$$

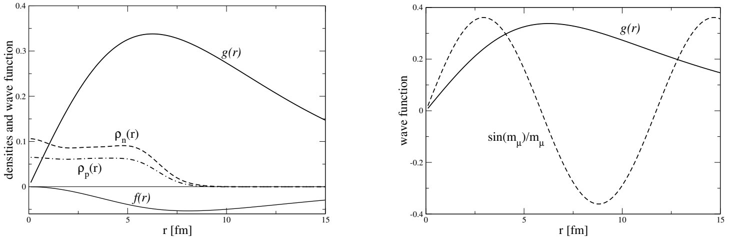


Figure: Radial muon wave function and nuclear densities of ^{208}Pb. Left panel: large ($g(r)$) and small ($f(r)$) components of the muon wave function, together with the proton and neutron densities of ^{208}Pb as a function of the radius; Right panel: large component of the muon wave function $g(r)$ compared to normalised $\sin(m_\mu r)/m_\mu$.

Coherent and non-coherent transitions

The non-coherent part of the probability $\Gamma_{i\to f}$ is given by

$$M^2_{gs\longrightarrow exc} = \sum_f \left(\frac{p_{e_f}}{m_\mu}\right)^2 M^2_{\mathrm{Nucl}}(J_f^\pi),$$

From the effective Lagrangian only the scalar and vector terms contribute to the squared matrix element of the coherent process:

$$M^2_{\mathrm{gs}\longrightarrow\mathrm{gs}} = (3g_V f_V)^2 \left[\left(1+\frac{1}{3}\beta\right) ZF_Z + \left(1-\frac{1}{3}\beta\right) NF_N\right]^2,$$

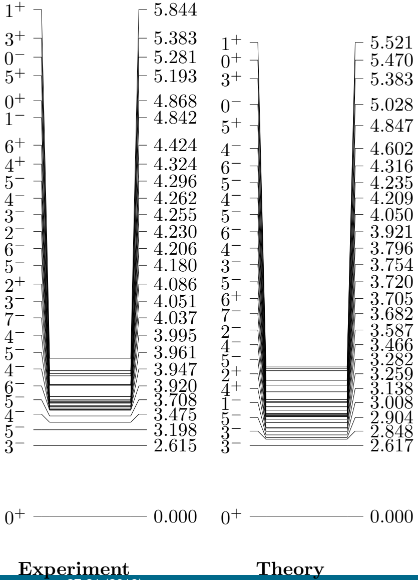
with the proton and neutron nuclear form factors F_Z and F_N given by the expressions

$$F_Z = \frac{1}{Z}\sum_j (2j+1)\langle j||j_0(|\mathbf{p_e}|r)||j\rangle (V_j^Z)^2,$$

$$F_N = \frac{1}{N}\sum_j (2j+1)\langle j||j_0(|\mathbf{p_e}|r)||j\rangle (V_j^N)^2,$$

Results

The spectrum and wave functions of Pb208



Results

Nuclear Matrix Elements: Exact muon wave function

	$q, q' = p, p'$	$q, q' = n, n'$	$q, q' = p, n'$
$\sum_f (p_{e_f}/m_\mu)^2 m_e \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	0.445×10^{-2}	0.387×10^{-2}	0.261×10^{-3}
$\sum_f (p_{e_f}/m_\mu)^2 E_{e_f} \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	0.716	0.660	0.615×10^{-1}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	0.315×10^{-1}	0.308×10^{-1}	0.360×10^{-2}
$\sum_f (p_{e_f}/m_\mu)^2 m_e \mathcal{M}_{\langle \sigma \rangle}(qq')$	0.455×10^{-2}	0.400×10^{-2}	-0.381×10^{-4}
$\sum_f (p_{e_f}/m_\mu)^2 E_{e_f} \mathcal{M}_{\langle \sigma \rangle}(qq')$	0.719	0.675	-0.116×10^{-1}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \sigma \rangle}(qq')$	0.311×10^{-1}	0.310×10^{-1}	-0.762×10^{-3}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \Lambda \rangle}(qq')$	-0.924×10^{-4}	0.362×10^{-4}	-0.206×10^{-3}

Table: Nuclear matrix elements, of the orbital, spin, and tensor operators In all of these matrix elements the exact muon wave function is used

Results

Approximated muon wave function

	$q,q'=p,p'$	$q,q'=n,n'$	$q,q'=p,n'$
$\sum_f (p_{e_f}/m_\mu)^2 m_e \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	0.675×10^{-2}	0.593×10^{-2}	0.306×10^{-3}
$\sum_f (p_{e_f}/m_\mu)^2 E_{e_f} \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	1.085	1.016	0.727×10^{-1}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \mathbf{1} \rangle}(qq')$	0.475×10^{-1}	0.473×10^{-1}	0.426×10^{-2}
$\sum_f (p_{e_f}/m_\mu)^2 m_e \mathcal{M}_{\langle \sigma \rangle}(qq')$	0.680×10^{-2}	0.603×10^{-2}	-0.588×10^{-4}
$\sum_f (p_{e_f}/m_\mu)^2 E_{e_f} \mathcal{M}_{\langle \sigma \rangle}(qq')$	1.074	1.018	-0.163×10^{-1}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \sigma \rangle}(qq')$	0.462×10^{-1}	0.468×10^{-1}	-0.103×10^{-2}
$\sum_f (p_{e_f}/m_\mu)^2 p_{e_f}^2/(2M_N) \mathcal{M}_{\langle \Lambda \rangle}(qq')$	-0.124×10^{-3}	0.176×10^{-4}	-0.213×10^{-3}

Table: Idem as the previous table with the replacement of the radial factor $g(r)j_l(p_e r)$ by $g(R)j_l(p_e r)$

Non-coherent process

term	total	spin-independent	spin-dependent	spin-independent/total (%)	spin-dependent/total(%)
scalar	2.217	2.217	0.	100	0.
vector	2.013	2.013	0.	100	0.
axial-vector	1.850	0.	1.850	0	100
tensor	2.032	0.005	2.027	0.2	99.8

Table: Contribution of different terms of the effective Lagrangian to the total squared matrix element of the non-coherent process (2nd column), its spin-independent (3rd column), and spin-dependent (4th column), parts. The 5th and 6th columns show the percentages of spin-independent and spin-dependent terms to the total squared matrix element. All values are given in units of G_F^2

The non-coherent N.M.E.of the W-exchange and SUSY models

model	total	spin-independent	spin-dependent	spin-independent/total (%)	spin-dependent/total(%)
W-exchange	3.532	2.691	0.841	76.2	23.8
SUSY	34.331	5.024	29.307	14.6	85.4

Table: Contribution of W-exchange and SUSY models to the total squared matrix element of the non-coherent process (2nd column), its spin-independent (3rd column) and spin-dependent (4th column) parts. The 5th and 6th columns show the percentages of spin-independent and spin-dependent terms to the total squared matrix element.

Multipole contributions to the non-coherent process

J^π	W-exchange			SUSY		
	total	spin-independent	spin-dependent	total	spin-independent	spin-dependent
0^-	0.056	0.000	0.056	2.110	0.000	2.110
0^+	0.636	0.636	0.000	1.087	1.087	0.000
1^-	1.217	1.074	0.143	7.134	2.218	4.916
1^+	0.253	0.000	0.253	8.238	0.000	8.238
2^-	0.166	0.000	0.166	5.809	0.000	5.809
2^+	0.735	0.653	0.082	3.934	1.099	2.835
3^-	0.298	0.279	0.019	1.270	0.504	0.766
3^+	0.090	0.000	0.090	3.364	0.000	3.364
4^-	0.025	0.000	0.025	0.920	0.000	0.920
4^+	0.045	0.042	0.003	0.257	0.103	0.155
5^-	0.005	0.005	2×10^{-4}	0.027	0.013	0.014
5^+	0.004	0.000	0.004	0.181	0.000	0.181

Table: Contributions of various multipoles (J^π) to the total squared matrix element of the non-coherent process, for the W-exchange model and for the SUSY model

Multipole decomposition of the non-coherent channel (W-exchange and SUSY models)

model	$M^2_{gs \longrightarrow gs}$	$M^2_{gs \longrightarrow exc}$
W-exchange	1738.8	3.5
SUSY	2904.3	34.3

Table: Square matrix elements for coherent and non-coherent transitions for the W-exchange and SUSY models.

Multipole contributions (histograms)

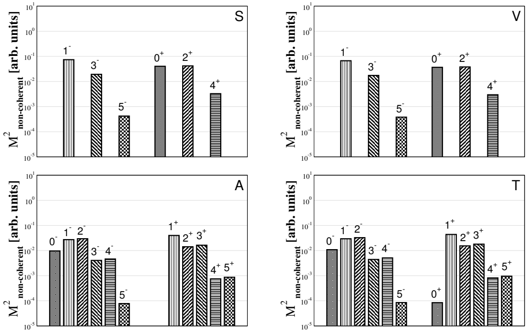
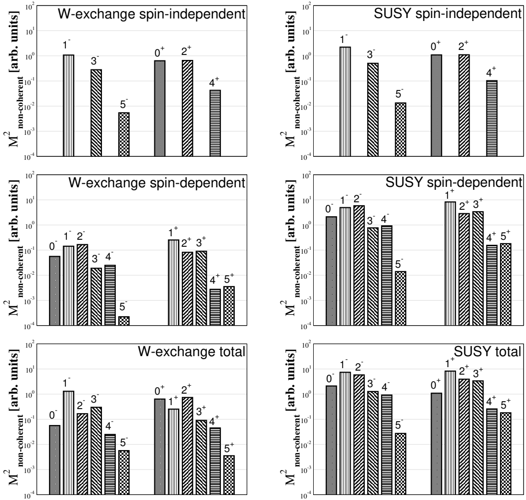


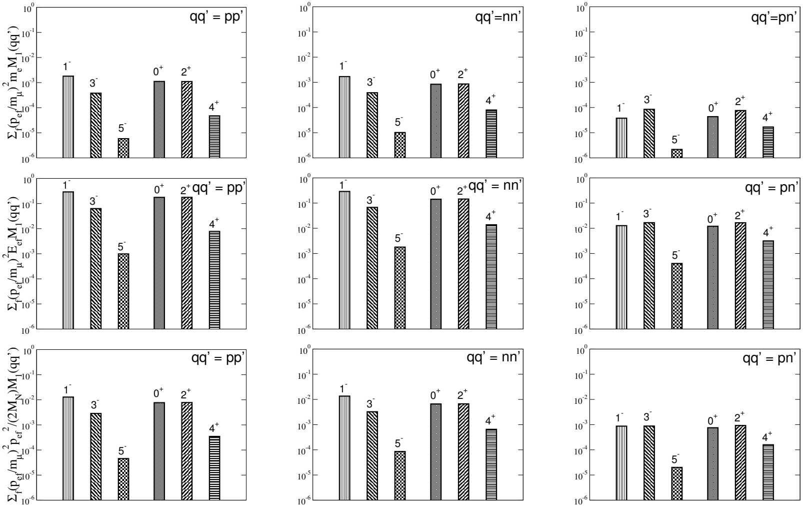
Figure: Multipole decomposition of the squared matrix element of the non-coherent process for different terms of the effective Lagrangian, namely: S (scalar terms), V (vector terms), A (axial-vector terms), and T(tensor terms), respectively. All values are nuclear matrix elements square, regardless of the specific model for the ($\mu^- \longrightarrow e^-$) conversion.

W-exchange and SUSY models
(histograms) for the non-coherent case



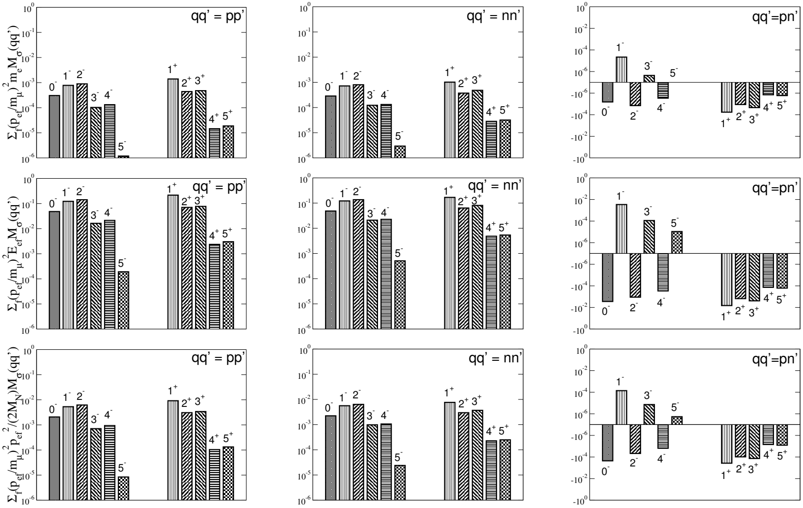
Non-coherent channel

spin-independent n.m.e, proton and neutron contributions, independently of the
model



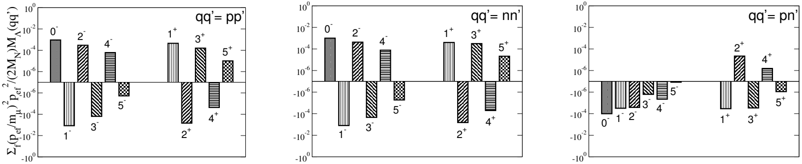
Non-coherent channel

spin-dependent n.m.e, proton and neutron contributions, independently of the model



Non-coherent channel

tensor-type n.m.e, proton and neutron contributions, independently of the LFV mod



Conclusions

The theoretical approximations leading to the LFV transition amplitudes are dependent on the assumptions made to approximate the sub-nucleonic and nucleonic parts of the currents and ultimately on the form of the adopted interactions to describe nuclear correlations, as well as on the specific form of the Lagrangian. While the nuclear part of the calculations may be tested by other means, for instance by using the same wave functions to calculated electromagnetic transitions, decays and particle transfer reactions on the nucleus host of the LFV process, the particle physics part of it is still open to schemes which goes beyond the standard model of the electroweak transitions.

Then, the task to determine the influence of the underlying LFV model in setting up limits on the observability of $(\mu^- \longrightarrow e^-)$ conversion, may be facilitated by the use of the multipole decomposition of the nuclear response as we have done in this work.